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Stability analysis of discrete dynamical system using khan-iteration technique

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Abstract
In the latest disciplines of research, controlling of chaos became the most important topic of research in non-linear dynamical systems which have large number of applications in engineering and scientific areas. In this reference, our objective is to develop controlling system by using the Picard-Mann hybrid procedure known as Khan-iteration. The chaotic region occur in dynamical systems due to the inconsistent periodic and fixed orbits of the system. Hence, the irregular periodic and fixed states are stabilized using control mechanism. Moreover, some theoretical and experimental results are observed which are followed by examples and theorems. At last, we examined the unstable states of the K- system and stabilized them by using Lyapunov exponent.

Keywords: Stability, Lyapunov exponent, K- system, periodic and fixed states and controlling chaos

Introduction

In nonlinear dynamical systems, the word "chaos" has attracted the many of authors and researchers to do research in this field. Differential equations are obtained through the changes in the environmental conditions and the discrete dynamical system will be considered as the difference equation. In dynamical systems, the theory of chaos controlling has broad range of applications in many branches like in traffic flow system, laser technology, image cryptography etc. Chaos in nonlinear systems occurred due to unstable fixed and periodic points, then it can be controlled by using iteration mechanisms. Today, there are so many techniques for controlling chaos. The very first discussion on chaos controlling was given by Ott. *et al*. [24] through feedback procedure in which the parameters are adjusted to get stabilized the system. In 1991, Ditto *et al*. [14] investigated the power method for controlling chaos of a system having unstable periodic orbits of order one and two. De Vieira *et al*. [33] demonstrated a method of controlling in which delayed feedback is used to the unstable orbit to stabilize them. Afterward, many more techniques were discovered for controlling chaos by using different iteration feedback such as by proportional pulses, through growth rate adjustment, oscillating feedback technique and controlling chaos by using predictive control [13, 16, 20, 26, 27]. In 2009, V. Berinde [9] described a systematic and straightforward method for stabilizing the discrete dynamical system. One may also refer to [1, 10, 28].

Stabilization of unstable orbits of a discrete dynamical system through using fixed point iteration method, has became the most important topic of research area. That is why, so many authors attracted towards this concept and theorized about this. The fixed point iteration techniques are applicable in many disciplines of technology and engineering. New techniques of stabilization of unstable orbits is discovered by P. Carmona [12] in 2011, using constant proportional feedback and then a method of proportional feedback control with pulses was discussed by E. Braverman [11] in 2012. J. Du *et al*. [15] described a method of controlling chaos that was applicable on an economic dynamical system. Moreover, the stability of the standard logistic map by using Ishikawa iteration [17] technique is studied by B. Parsad and K. Katiyar [25]. Later, a discrete fractional logistic map and fractional delayed logistic map was given by Baleanu and Wu [34-35] and then stability of Caputo like discrete fractional systems is analyzed by Baleanu *et al*. [8] in 2017. M. A. Khan *et al*. [19] proposed a technique by using nonlinear control for Lorenz like system to a dynamical system. In that year, for controlling the chaos for Chau's circuit, a steady feedback control is established by M. Mukherjee and S. Halderb [23]. Next year, Ashish *et al.* [4-5] examined the dynamical properties of the conventional map through superior approach and also demonstrated their applications in traffic model. Then they observed the controlling of chaos via superior feedback and also described their applications in traffic models. In year 2022, Sanjeev *et al.*, Monika *et al.* and Ashish *et al.* [6, 22, 29] described chaotic characteristics of the standard equation through Euler's Algorithm, discrete difference map using Mann iteration [21] and Sine logistic map through Picard orbit respectively.

Latterly, in the year of 2023, many more new techniques and methods are demonstrated by authors for controlling chaos. There are a series of papers in which authors described the way of stabilization of unstable fixed and periodic states. D. Sekman and V. Karakaya [30-32] established different procedures for stabilization of dynamical systems using S- Iteration, Ishikawa- Iteration and multistep iteration process. Ashish *et al.* [3] described the method of controlling chaos in which there are four control parameters and it is known as Noor control mechanism. It gives stability rate faster than any of existing methods. A discrete composition of two maps: Euler's and Logistic map studied by Ashish and J. Cao [2]. They discussed their scaling properties like bifurcation scale, fork-width scale. Recently, in January 2024, Ashish and M. Sajid [7] investigated a procedure of hybrid control chaos.

In this article, we use a Picard- Mann hybrid iterative process for the controlling of chaos in one-dimensional discrete dynamical system. In this paper, firstly we introduced whole about chaos and controlling chaos and a brief literature view which is put in section 1st. Then, the main concept of controlling using theorems and examples are illustrated under section 2nd. In section 3rd, Lyapunov exponent behavior is discussed briefly with examples and last the results are concluded in section 4.

Controlling chaos by using Khan Iteration process

The equation of uni-dimensional distinct dynamical system may be written in general form as,

$$x_{n+1} = \varphi(x_n), \tag{2.1}$$

Where $n \in N$ and $\varphi(x)$ is a function from [a, b]→ [a, b] and. This defined function should have a fixed point in interval [a, b]. Now, the original system can be transformed into K-system by applying the definition of Khan Iteration on equation (2.1) as,

$$x_{n+1} = K(\alpha, \varphi) = \varphi[(1-\alpha)x_n + \alpha\varphi(x_n)] \tag{2.2}$$

Where $\alpha \in (0, 1)$ is the control parameter. Now, here we will explain the process of stability of unstable fixed points by control system generated by Khan Iteration.

Khan - Iteration [18]: Let $T: C \to C$ be a function and suppose $\{x_n\}$ and $\{y_n\}$ be the sequences from C, then.

$$x_{n+1} = T(y_n),$$

$$y_n = (1-\alpha_n)x_n + \alpha_n T(x_n),$$

Where $n \in N$ and $\alpha_n \in (0,1)$

Now, we explain the control mechanism by some theorems.

Theorem 2.1: Let us suppose, the irregular fixed point of the primary system (2.1) be x^*. If $\varphi(x^*) = x^*$, then $K(\alpha, \varphi)(x^*) = x^*$ for all $x \in [a, b]$. Then for the control parameter α there is always an effective regime for $K(\alpha, \varphi)$ in such a way,

$$|K'(\alpha, \varphi)(x^*)| < 1$$

for $|\varphi'(x^*)| \neq 1$.

Proof: Suppose the fixed point of primary system is x^* such that $\varphi(x^*) = x^*$, then the for the K- system the fixed point is.

$$K(\alpha, \varphi)(x^*) = \varphi[(1-\alpha)x^* + \alpha\varphi(x^*)]$$

$$= \varphi[(1-\alpha)x^* + \alpha x^*]$$

$$= \varphi(x^*)$$

$$= x^*$$

Hence, x^* is observed as a stationary point of $K(\alpha, \varphi)$.

Now, let this x^* is the irregular stationary point of the system $\varphi(x)$, then there will be $|\varphi'(x^*)| > 1$. Here, two case arises, one is $\varphi'(x^*) < -1$ and other is $\varphi'(x^*) > 1$. When $\varphi'(x^*) > 1$, the obtained stability range can't meet the definition range of parameter α and hence we determine the stability range only for one case $\varphi'(x^*) < -1$. Under this condition, we have two cases for $K(\alpha, \varphi)$.

Case I. For $\varphi'(x^*) < -1$, when $K'(\alpha, \varphi)(x^*) < 1$, then

$K'(\alpha,\varphi)(x^*) = \varphi'[(1-\alpha)x^* + \alpha\varphi(x^*)][(1-\alpha) + \alpha\varphi'(x^*)]$

$= \varphi'[(1-\alpha)x^* + \alpha x^*][(1-\alpha) + \alpha\varphi'(x^*)]$

$= \varphi'(x^*)((1-\alpha) + \alpha\varphi'(x^*))$

Now,

$\varphi'(x^*)\big((1-\alpha) + \alpha\varphi'(x^*)\big) < 1$

$1-\alpha+\alpha\varphi'(x^*) < \frac{1}{\varphi'(x^*)}$

$\alpha(\varphi'(x^*)-1) < \frac{1}{\varphi'(x^*)} - 1$

$\alpha(\varphi'(x^*)-1) < \frac{1-\varphi'(x^*)}{\varphi'(x^*)}$

$\alpha < -\frac{1}{\varphi'(x^*)} = \alpha_{max}$

Case II. For $\varphi'(x^*) < -1$, when $K'(\alpha,\varphi)(x^*) > -1$

$\varphi'(x^*)\big((1-\alpha) + \alpha\varphi'(x^*)\big) > -1$

$1-\alpha+\alpha\varphi'(x^*) > -\frac{1}{\varphi'(x^*)}$

$\alpha(\varphi'(x^*)-1) > \frac{-1}{\varphi'(x^*)} - 1$

$\alpha(\varphi'(x^*)-1) > -\frac{1+\varphi'(x^*)}{\varphi'(x^*)}$

$\alpha > \frac{1+\varphi'(x^*)}{\varphi'(x^*)(1-\varphi'(x^*))} = \alpha_{min}$

Thus, the obtained interval is $(\frac{1+\varphi'(x^*)}{\varphi'(x^*)(1-\varphi'(x^*))}, -\frac{1}{\varphi'(x^*)})$. This is the interval for control parameter, where the irregular stationary points get stabilized.

Now, let's take some examples which illustrates the above theorems.

Example 2.2: Let's suppose the original system be the standard logistic map,

$x_{n+1} = \varphi(x_n) = 4x_n(1-x_n),$

This system has fully chaotic behaviour at r = 4 and has unstable fixed points $x_1 = 0$ and $x_2 = 3/4$ with $\varphi'(x_1) = 4 > 1$ and $\varphi'(x_2) = -2 < 1$. Determine the effective regime for parameter α, for stabilise these unstable fixed points by using K- system control system.
Solution: First of all, we evaluate the control system using logistic map as,

$K(\alpha,\varphi)(x_n) = \varphi[(1-\alpha)x_n + \alpha\varphi(x_n)]$

$= \varphi[(1-\alpha)x_n + \alpha 4x_n(1-x_n)]$

$= \varphi[x_n - \alpha x_n\alpha + 4\alpha x_n - 4\alpha x_n^2]$

$= \varphi(x_n(1+\alpha(3-4x_n))$

$= 4x_n(1+\alpha(3-4x_n)(1-x_n(1+\alpha(3-4x_n))$ (2.3)

Now, by using theorem 2.1, the effective regime for $x_1 = 0$, is

$= (-\frac{5}{12}, -\frac{1}{4})$

Which is a negative region, hence it can be neglected. Now, for $x_2 = \frac{3}{4}$,

$= (\frac{1}{6}, \frac{1}{2})$

We get an effective region for $x_2 = \frac{3}{4}$, in which this unstable point can be stabilize. For α = 0.2, 0.3,0.4 and 0.5, we can obtain the stable solution, as it is displayed in figure 4. Figure 2 displays the functional plot for equation (2.3), for α = 0.2, 0.3, 0.4 and 0.5.

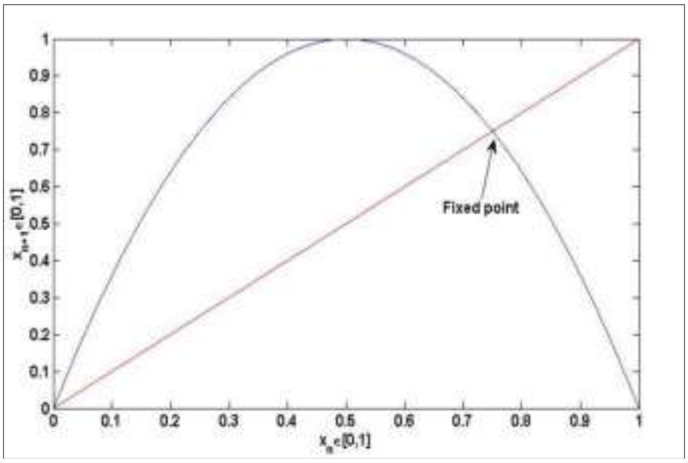

Fig 1: Functional plot of the original system $\varphi(x)$

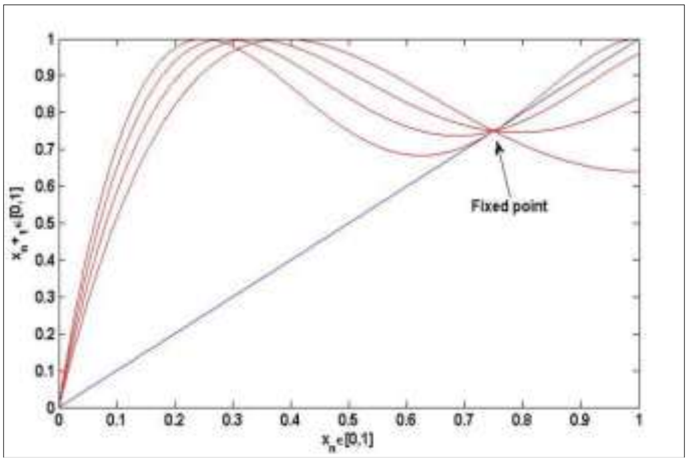

Fig 2: Functional plot of the K-system for α = 0.2, 0.3, 0.4 and 0.5

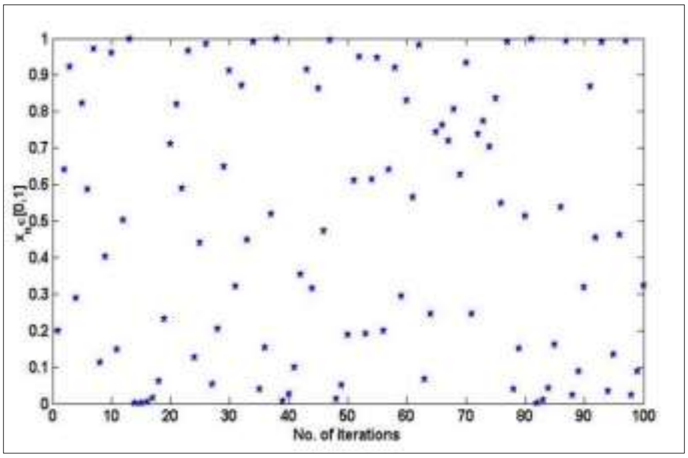

Fig 3: Unstable state for the original system $\varphi(x)$ for $x_0 = 0.4$

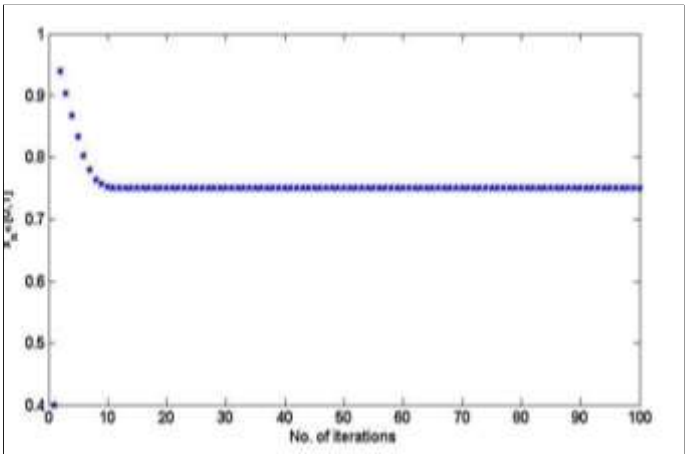

Fig 4: Stable solution for the K-system for $x_0 = 0.4$ and α = 0.4

Example 2.3: Let us consider the original system be $\varphi(x) = x(4x-3)^2$. Given equation have three fixed points x_1= 0, x_2 = 1 and x_3 = 1/2 with $\varphi'(x_1) = 9$, $\varphi'(x_2) = 9$ and $\varphi'(x_3) = -3$, it is observed that all three fixed points are unstable. Then, determine an effective region for parameter α such that these points can be stabilize.

Solution: Here, we can find the effective interval for control parameter by using theorem 2.1, for the unstable fixed point x_3 = 1/2 as it has $\varphi'(x_3) = -3 < 1$.

Now, from theorem 1.1, effective regime is $(\frac{1+\varphi'(x^*)}{\varphi'(x^*)(1-\varphi'(x^*))}, -\frac{1}{\varphi'(x^*)})$

Thus, we get effective regime as (1/6, 1/3).

2.3: Controlling chaos in unstable periodic states using Khan Iteration

In this part, we examine about unstable periodic phase of the original system and stabilize them by using K- system. Here, first we define the original system by replacing $\varphi(x)$ by $\varphi^p(x)$, we define K- system as,

$$K_p(\alpha, \varphi) = \varphi^p((1-\alpha)x_n + \alpha\varphi^p(x_n) \qquad (2.4)$$

where x ∈[a, b] and $\varphi^p(x)$ is the pth recurrent of an original system. Now, here we examine that the original system and K-system has the same set of unstable fixed points but we have to find an effective interval for unstable fixed points using K-system. Now, we discuss results by using theorems.

Theorem 2.4: Consider that the irregular periodic point of order-p of the system (2.1) be x^*. Then, if $\varphi^p(x^*) = x^*$ then, $K_p(\alpha, \varphi)(x^*) = x^*$. If it so, then there always exits an effective interval for parameter α in the K-system such that,

$|K_p{}'(\alpha, \varphi)(x^*)| < 1$ for $|\varphi^{p\prime}(x^*)| \neq 1$.

Proof: We have given that the irregular periodic point for the system of order-p is x^*. that is, $\varphi^p(x^*) = x^*$ and we have to show that K- system has same set of periodic points. Now,

$$K_p(\alpha, \varphi)(x^*) = \varphi^p((1-\alpha)x^* + \alpha\varphi^p(x^*)$$

$$= \varphi^p((1-\alpha)x^* + \alpha x^*$$

$$= \varphi^p(x^*)$$

$$= x^*$$

Hence, it proved that K-system has same periodic points as of primary system.
Now, consider the x^* be the point of the original system that is unstable periodic point of order-p. Then, using control procedure of K- system, the effective interval can be obtained by solving, $|K_p{}'(\alpha, \varphi)(x^*)| < 1$, where.

$$K_p{}'(\alpha, \varphi)(x^*) = \varphi^{p\prime}((1-\alpha)x^* + \alpha\varphi^{p\prime}(x^*)$$

By solving this inequality, we obtain the effective regime interval as,

$$\left(\frac{1+\varphi^{p\prime}(x^*)}{\varphi^{p\prime}(x^*)(1-\varphi^{p\prime}(x^*))}, -\frac{1}{\varphi^{p\prime}(x^*)}\right]. \tag{2.5}$$

Now, we study the following example of the logistic equation.

Example 2.5 Let us define the logistic system φ: [0,1] → [0,1].

$$x_{n+1} = \varphi(x_n) = 4x_n(1-x_n)$$

Then, find out the stability interval for the irregular periodic point of order-2 by using control system of Khan iteration.

Solution: To find the periodic points of logistic map, we have to solve the equation given as,

$$\varphi^2(x^*) = \varphi(4x^*(1-x^*) = 16x^*(1-x^*)(4x^{*2} - 4x^* + 1).$$

By evaluating this, we get period-2 fixed points, $x_1^* = (5-\sqrt{5})/8$ and $x_2^* = (5+\sqrt{5})/8$. Both of these order-2 points are unstable as $\varphi^{2\prime}(x_1^*) = \varphi^{2\prime}(x_2^*) = -4 < -1$.
By using K- system, we get

$$K_2(\alpha, \varphi)(x^*) = \varphi^2((1-\alpha)x^* + \alpha\varphi^2(x^*)$$

According to equation (2.5), we get an effective interval for period-2 fixed points as, (3/20, 1/4].

3. Lyapunov Exponent for Khan-Iteration: In this part, our motive is to examine the Lyapunov exponent nature of the orbits of one-dimensional map by using Khan- Iteration system. Lyapunov exponent is known as the powerful feature that measures the sensitivity between two orbits. It evaluates the rate of convergence for the stable orbits and determines the rate of divergence for irregular behavior. Now, for the controlled system, we have to find out the Lyapunov exponent, $K(\alpha, \varphi) = \varphi[(1-\alpha)x + \alpha\varphi(x)]$, where $\varphi(x)$ is the primary system. Now, we assume two starting points for orbits in K- system are y and $y + \varepsilon$, where $\varepsilon \in (0,1)$. In addition of this, let we consider γ denotes divergence rate between orbits and $\varepsilon e^{n\rho}$ represents the rate of exponential growth, where ρ denotes Lyapunov exponent. Now,

$$K(\alpha, \varphi)(y+\varepsilon) - K(\alpha, \varphi)(y) = \gamma,$$

and $K_n(\alpha, \varphi)(y+\varepsilon) - K_n(\alpha, \varphi)(y) = \varepsilon e^{n\rho}$,

$$\frac{K_n(\alpha,\varphi)(y+\varepsilon) - K_n(\alpha,\varphi)(y)}{\varepsilon} = e^{n\rho}$$

Now, taking the limit ε →0, in the above equation, we get

$$\lim_{\varepsilon\to 0} \frac{K_n(\alpha,\varphi)(y+\varepsilon) - K_n(\alpha,\varphi)(y)}{\varepsilon} = e^{n\rho}$$

Now, we can write this as,

$$K'_n(\alpha, \varphi)(x) = e^{n\rho} \quad (3.6)$$

Now, applying Logarithm on both side, we get.

$$\log|K'_n(\alpha, \varphi)(x)| = n\rho,$$

$$\frac{1}{n}\log|K'_n(\alpha, \varphi)(x)| = \rho$$

where $K'_n(\alpha, \varphi)(x)$ is the derivative of $K_n(\alpha, \varphi)(x)$, $\alpha \in (0,1)$ and $n \in N$. Now, for the derivation of nth degree polynomial, we use the chain rule as follows, from equation (3.6),

$$|K'_n(\alpha, \varphi)(x)| = K'(\alpha, \varphi)(x_1).K'(\alpha, \varphi)(x_2).K'(\alpha, \varphi)(x_3)\ldots\ldots\ldots\ldots K'(\alpha, \varphi)(x_n) = e^{n\rho}$$

After applying logarithm on both sides, we get

$$\rho = \frac{1}{n}\sum_{i=1}^{n}\log|K'(\alpha, \varphi)(x_i)|. \quad (3.7)$$

Now, we can calculate Lyapunov exponent easily by using equation (3.7). The regularity and irregularity of periodic and stationary orbits can also determined by using Lyapunov exponent, that is, if Lyapunov exponent is positive, then the orbits are unstable and if value of Lyapunov exponent is negative, then orbits are stable.
Here, we discuss some examples of Lyapunov exponent, which deciding the stability or instability of orbits of K- system.

Example 3.1: For the uni-dimensional standard equation $\varphi(x) = 4x(1-x)$, measure the Lyapunov exponent of the fixed point $x = 3/4$ and for $\alpha = 0.2$, by using the K-system $K(\alpha, \varphi)(x) = \varphi[(1-\alpha)x + \alpha\varphi(x)]$.

Solution: We have the control system, $K(\alpha, \varphi)(x) = \varphi[(1-\alpha)x + \alpha\varphi(x)]$, then we get.

$$K'(\alpha, \varphi)(x) = \varphi'(x)((1-\alpha) + \alpha\varphi'(x))$$

where $\varphi'(x) = 4 - 8x$

Now, we get $K'(\alpha, \varphi)(x) = (4-8x)((1-\alpha) + \alpha(4-8x))$ (3.8)

$$= (4 - 8 \times 3/4)((1-0.2) + 0.2(4 - 8 \times 3/4))$$

$$= (-2)((0.8) + 0.2(-2))$$

$$= -2 \times 0.4$$

$$K'(\alpha, \varphi)(x) = -0.8$$

then, by using equation (3.7)

$$\rho = \log|K'(\alpha, \varphi)(x)|$$

$$= \log|-0.8|$$

$$= -0.09691 < 0$$

Hence, we observed a negative Lyapunov exponent for fixed point $x = 3/4$ and for $\alpha = 0.2$, hence, it is stable point.

Example 3.2: Calculate Lyapunov exponent value for the periodic points of the standard equation under K- system for $\alpha = 0.25$ and points are $x_1 = \frac{5-\sqrt{5}}{8}$ and $x_2 = \frac{5+\sqrt{5}}{8}$.

Solution: We have K-system, $K(\alpha, \varphi)(x) = \varphi[(1-\alpha)x + \alpha\varphi(x)]$

Now, by putting $x_1 = \frac{5-\sqrt{5}}{8}$ and $x_2 = \frac{5+\sqrt{5}}{8}$ in equation (3.8), we get

$$K'(\alpha, \varphi)(x_1) = (4 - 8 \times \frac{5-\sqrt{5}}{8})((1-\alpha) + \alpha(4 - 8 \times \frac{5-\sqrt{5}}{8}))$$

$$= (-1+\sqrt{5})((1-0.25) + 0.25(-1+\sqrt{5}))$$

$= 1.3089$

$K'(\alpha,\varphi)(x_2) = (4 - 8 \times \frac{5+\sqrt{5}}{8})((1-\alpha) + \alpha(4 - 8 \times \frac{5+\sqrt{5}}{8}))$

$= (-1-\sqrt{5})((1-0.25) + 0.25(-1-\sqrt{5}))$

$= -3.2360 \times (0.75 + 0.25 \times -3.2360)$

$= -3.2360 \times -0.059$

$= 0.1909$

Now, from equation (3.7), we get

$\rho = \frac{1}{2}\sum_{i=1}^{2} \log|K'(\alpha,\varphi)(x_i)|$

$= \frac{1}{2}(\log|K'(\alpha,\varphi)(x_1)| + \log|K'(\alpha,\varphi)(x_2)|)$

$= \frac{1}{2}(\log|1.3089| + \log|0.1909|)$

$= \frac{1}{2}(0.1169 + (-0.7191))$

$= -0.6022/2 = -0.3011$

$\rho = -0.3011 < 0$

Hence, we find that the it is negative, which implies that the periodic orbits of order-2 are stable.

Remark 3.3: It is observed that for the logistic equation, as original system has chaotic behavior for r = 4 while in the control system of Khan -iteration, the unstable fixed and periodic points are stabilized using K-system.
In this K-system, we found that the original system can be stabilize for r > 4 also for some parametric value α. Here, we can notice from the given table that, there is different value for parameter r for different values of parameter α.

Table 1: Maximum Stability Threshold (r_{max}) for Parameter α in the K-system

Parameter α	r_{max}
0.2	4.637
0.3	4.2
0.4	4.037

The whole spectrum for the L. E. for original system is display in figure 5, which have chaotic region for r ∈ [3.55, 4] while the plot of L. E. for the K-system for α = 0.2 in figure 6 and we noticed that there is completely stability of fixed points and have negative value of Lyapunov exponent for all x ∈ [0, 1]. In figure 7, we display the comparative graph of Lyapunov exponent for original and K-system.

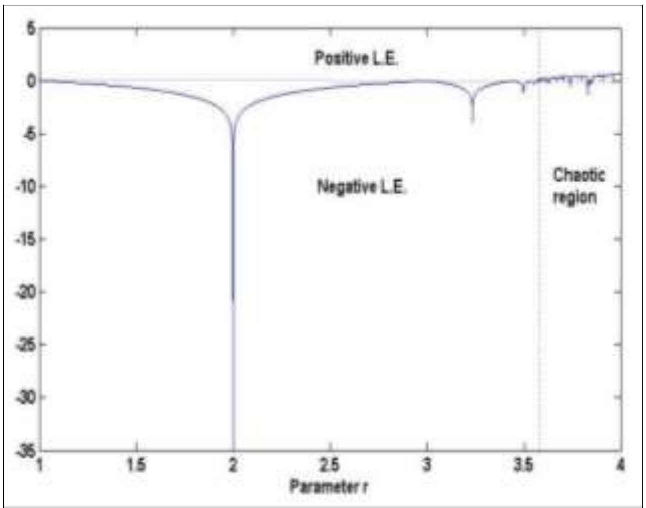

Fig 5: L.E. plot for the original system for r ∈[1, 4]

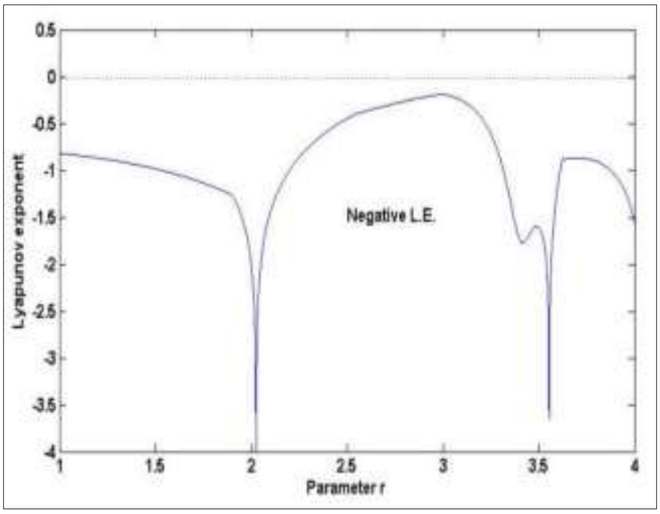

Fig 6: L.E. plot for the K-system for r ∈[1, 4] and α = 0.2

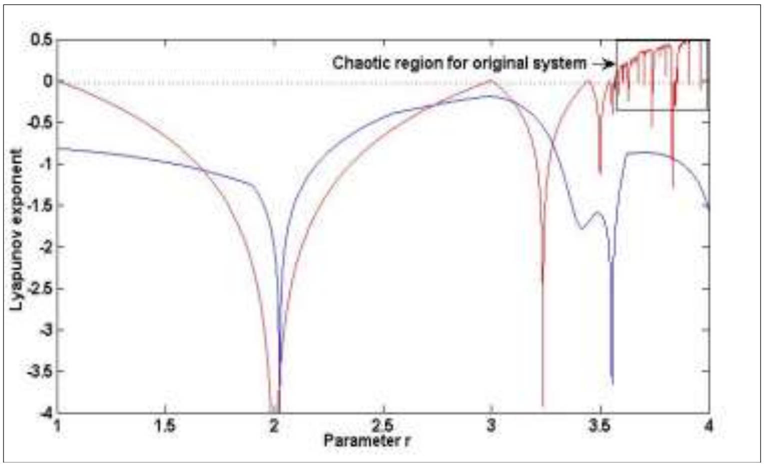

Fig 7: Comparative L.E. plot for original and controlled system

Conclusion

In this study, firstly we examined the one-dimensional maps. The characteristics of stationary and periodic points of the uni-dimensional map is investigated carefully. A method is discussed for controlling chaos in the nonlinear maps. All through this study, a two step technique which is known as Khan iteration, is applied for regulating the unstable stationary points of the original system. Here, some theoretically and experimentally results are discussed:

1. Firstly, for the one-dimensional maps, the common characteristics are examined by using Khan iteration method and we have taken two maps as an example, famous logistic map and one is the cubic map $x(4x-3)^2$. Moreover, theorem 1.1 is established for obtaining the control region for parameter α to control the instability of the unstable stationary points of the primary system.
2. Theorem 2.2 explained the periodic behavior of the one- dimensional map using K-system and the interval in which stationary points can be stabilized for the parameter α is examined for the logistic map $4x(1-x)$, discussed in example 2.4 briefly.
3. Lyapunov exponent behavior is also studied for the original system and controlled system. Example 3.1 and 3.2, indicates briefly the L.E. value of the periodic and stationary points for the logistic system.
4. Completely Lyapunov exponent nature of the original and controlled procedure is displayed using graphs in figure 5 and 6, and observed the value of growth rate parameter $r > 4$, for some particular values of α.

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