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Revisiting assignment problem: A novel approach to optimality

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Theme
Introduction of a novel and faster approach to optimal solution of assignment problem.

Abstract
Known since decades, all pursue standard algorithm to arrive at optimal basic feasible solution of assignment problem. Retaining the prime objective of assignment, we like to share what our approaches are that initially searches for a basic feasible solution and attempts to rectify each allocation to its optimal value. We, strongly, are of the opinion that this novel approach possesses logical but faster proceedings to optimality: once on achieving basic feasible solution. We are of the opinion that this new approach, on the basis of its own merits, will be widely accepted in the academic world of the subject.

Keywords: Assignment problem, column minimum, Jha method
[*Method's designation honours researchers surname]

1. Introduction
This section narrates some salient features of Hungarian algorithm which targets minimization of sum total of allocation values of one to one assignment.

1.1) Mathematical Facts
We have, in this case, an assignment matrix $\boldsymbol{A = (a_{ij})}$; $\boldsymbol{i, j \text{ is an integer from } 1 \text{ to } m}$.
Let $\boldsymbol{i \text{ and } j}$ stand for each one of $\boldsymbol{m}$ entries for rows $\boldsymbol{R_1, R_2, R_3, \ldots\ldots\ldots, R_m}$ and $\boldsymbol{C_1, C_2, C_3, \ldots\ldots\ldots, C_m}$ for rows and columns respectively. This suggests that the assignment matirx is a square matrix of order $\boldsymbol{m \times m \text{ and there are } m^2}$ cells. Let each cell value be called $\boldsymbol{C_{ij}}$ for each $\boldsymbol{i \text{ and } j}$. We have for each $\boldsymbol{R_i}$ a unique $\boldsymbol{C_j}$ with $\boldsymbol{C_{ij}}$ - as an assignment value of corresponding $\boldsymbol{(i, j)^{th}}$ cell. A special case of Transportation problem calls for exactly $\boldsymbol{m}$ allocated cells. It is a case of highly degenerate solution in which instead of $\mathbf{m + m - 1 = 2m - 1}$allocations, we have exactly $\mathbf{m}$ allocations. Here we have the targeted objective that the sum total of all $\boldsymbol{C_{ij}}$;

$$\boldsymbol{i.e. z = \sum\sum C_{ij}} \tag{1}$$

$\boldsymbol{\text{for each } i \text{ there is exactly one } j - i, j \text{ from } 1 \text{ to } m}$; $\boldsymbol{\text{where } m \in N}$, must be minimum.

1.2) Operation Moded
Hungerian method is decades old and has standard procedural routine which after a finite number of iterative steps descends to an optimal basic feasible solution which is surely degenerate and may suggest multiple optimality. We have, after many rounds of logical attempts, concluded to a novel approach. This method first finds a basic feasible solution which may be optimal and if not, then on applying confirmatory tests of search and reshuffling tends to optimality.

1.3) About the new approach
As described earlier, we have $\boldsymbol{m}$ rows $\boldsymbol{R_1}$ to $\boldsymbol{R_m}$ and $\boldsymbol{m}$ columns designated as $\boldsymbol{C_1}$ to $\boldsymbol{C_m}$. Also we search for such cells $\boldsymbol{(R_i, C_j)/(i, j)}$ cell processing the assignment values $\boldsymbol{C_{ij}}$ [for one $\boldsymbol{i}$ value there is exactly one uniquely determined $\boldsymbol{- j \text{ and } j \in 1 \text{ to } m}$] so that the sum $\boldsymbol{z = \sum\sum C_{ij}}$ which is minimum.

1.3.1) General Format
In this section we describe the general format and keep on gradually adding, further aspects of the Jha algorithm.

Assignment Matrix

Row \ Column	C_1	C_2	$\cdots$	C_j	$\cdots$	C_m
R_1	C_{11}	C_{12}	$\cdots$	$\cdots$	$\cdots$	C_{1m}
R_2	C_{21}	C_{22}	$\cdots$	$\cdots$	$\cdots$	C_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_j	C_{j1}	C_{j2}	$\cdots$	$\vdots$	$\cdots$	C_{jm}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_m	C_{m1}	C_{m2}	$\cdots$	C_{mj}	$\cdots$	C_{mm}

2) Unveiling Jha's Method - Algorithm
Following paragraphs lays the format followed by the objective function given in section - 1 mathematical facts.

We find $\mathbf{m}$ such cells (R_i, C_j); i **and** $j \in 1$ **to** m such that $\mathbf{Z} = \sum\sum \mathbf{C_{ij}}$ is minimum. This set of m numbers of (R_i, C_j) pairs indicates optimal basic feasibility. Initially the first sweep gets the basic feasible solution and the second filter, by probable reallocation, settles down to optimality.

2.1) How does it work?
We begin towards the direction of finding a basic feasible solution. Each step towards the solution is parallely sounded by an illustration.

Step 1- Column Minimum

We find minimum $\mathbf{C_{ij}}$ value of each column and note the same below the horizontal line segment drawn underneath each column.

Step 2 - Set of Minimum Values

At the end of step 1, we have set of $\mathbf{m}$ values - Minimum sought out from each one of $\mathbf{m}$ columns. Arrange all these values in ascending order. Let this be the set

$$\mathbf{M_1} = \{\mathbf{m_1}, \mathbf{m_2}, \ldots\ldots, \mathbf{m_m}\};$$

Where $\mathbf{m_1} \leq \mathbf{m_2} \leq \ldots\ldots \leq \mathbf{m_m}$

Each $\mathbf{m_i}$ corresponds to exactly one column value (minimum) from all given columns.

Step 3 - Reformatting the matrix

Now we arrange all the columns in correspondence of the ascending order in coordination with members of the set

$$\mathbf{M_1} = \{\mathbf{m_1}, \mathbf{m_2}, \ldots\ldots, \mathbf{m_m}\}$$

The new format may look as follows:

R_i \ Rj	K_1	K_2	K_3	$\cdots$	K_j	K_m
R_1					m_i	
R_2		m_2				
R_3	m_1					
$\vdots$						
R_j						
$\vdots$						
R_m			m_3			
Column Minima						

[Note that each column $\mathbf{K_j}$ represents exactly one of the old columns.

e.g. column $\mathbf{K_1}$ may represents any one of the old columns $C_1, C_2, C_3, \ldots\ldots, C_m$.]

Step 4 - Search for column Minima and assign

Here in this section we proceed to find a basic feasible solution. Observe the first column $\mathbf{K_1}$ and find minimum entry ; mark it and assign it to that associated row - say $\mathbf{R_i}$; $\mathbf{i = 1}$ **to** $\mathbf{m}$. Delete or discard the particular row and the first column from the matrix.

Step 5 - Basic Feasible Solution

At the end, on completion of step - 4. we repeat the same procedure and move towards feasible solution.

Once the minimum is selected and concerned row and the first column (R_i, K_1) are discarded, we search for minimum entry from the remaining columns - in this case the second column.

Again find the set of minimums from all the columns.

Let the set of minimums be $M_2 = \{m_2, m_3, \ldots\ldots, m_m\}$

Arrange elements of this set M_2 in ascending order and rearrange the columns if required.

Now, find the minimum from the second column and mark it.

Also discard the second columns and the row say R_i $(i \neq j)$ corresponding to the minimum entry of the second column.

Keep on repeating the above routines for all the remaining columns.

Step 6: Identifying the BFS

At the end we have the matrix showing ordered minimum.

Enlist it is as follows

Assignment matrix

Row \ Column	K_1	K_2	K_3	$\cdots$	K_j	K_m	
R_1					m_i		
R_2		m_2					
R_3	m_1						
$\vdots$							m_m
R_j							
$\vdots$							
R_m			m_3				

This helps write the BFS

Resources	Column Entry	Cell Entry	
R_1	C_j	m_1	C_{1j}
R_2	C_2	m_2	C_{2j}
R_3	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_j	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_m	C_3	m_3	C_{3j}

Step 7 - Optimality Check

In this final step - 7, we show optimality check to find the optimal (minimum) sum of all the $\mathbf{m_i}$ **or** $\mathbf{C_{ij}}$ values for $\mathbf{i}$, $\mathbf{j} \in$ **1 to m,** so that the sum total reaches optimality. This was required from the equation - 1. In this section we show the procedure by logical illustration. consider, from the above list the highest of $\mathbf{C_{ij}}$ entry and other components as shown below.

Assignment matrix

R_1 / C	K_1	K_2	K_3 C	K_4	K_5
R_1		c_{12}		c_{14}	m_i
R_2					
R_3		c_{32}		c_{34}	
R_4					
R_5					

1. Say the highest $\mathbf{C_{ij}}$ entry is $\mathbf{c_{34}}$ - allocated cell.
2. Look for some minimum entry say $\mathbf{c_{32}}$ in that row. Find $|\mathbf{c_{34}} - \mathbf{c_{32}}| = \mathbf{L_1}$ (say)
3. There must be exactly one assignment in that column - say $\mathbf{c_{12}}$.
4. Now find corresponding entry of $\mathbf{c_{34}}$ in that cell - say $\mathbf{c_{14}}$
5. Find $|\mathbf{c_{12}} - \mathbf{c_{14}}| = \mathbf{L_2}$ (say)
6. If $\mathbf{L_1} > \mathbf{L_2}$ then interchange $\mathbf{c_{34}}$ - allocated cell value - with $\mathbf{c_{32}}$ and $\mathbf{c_{12}}$ - allocated cell value - with $\mathbf{c_{14}}$.
7. $\mathbf{L_2} \geq \mathbf{L_1}$ then look for next minimum in row entry of $\mathbf{c_{34}}$ and repeat the above steps.
8. Repeat the above procedure for remaining entries - most possibly some higher $\mathbf{c_{ij}}$ value to $\mathbf{c_{32}}$ value this may need the same treatment of reshuffling.

This will be clear by the illustration.

2.2) Illustration

We give an illustration to sound the above procedure. Consider the assignment matrix. We want to allocate some $\mathrm{c_{ij}}$ - column value to each row $\mathrm{R_i}$ so that the sum value of allocation is minimum.

C / R	C_1	C_2	C_3	C_4	C_5
R_1	5	8	7	5	4
R_2	3	2	7	6	9
R_3	8	1	5	7	4
R_4	10	12	8	7	6
R_5	9	12	10	8	5

Solution: Following step 1, we find column minima.

R / C	C_1	C_2	C_3	C_4	C_5
R_1	5	8	7	5	4
R_2	3	2	7	6	9
R_3	8	1	5	7	4
R_4	10	12	8	7	6
R_5	9	12	10	8	5
Column Minimum	3	1	5	5	4

1) Set $M_1 = \{1, 3, 4, 5, 5\}$ **which is *in the ascending order*** written from column minimum.
2) Rearrangement of columns
3) Mark minimum from first column and discard first column $(\mathbf{C_2})$ and third row $(\mathbf{R_3})$ so we the matrix as follows:

C / R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10
Column Minimum	-	3	4	5	7

4) In the same way mark column minima from second column - $\mathbf{C_1}$. **i. e.** Discard $\mathbf{R_2}$ and $\mathbf{C_1}$ as shown below

C / R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10
Column Minimum	-	-	4	5	7

$(\mathbf{R_2}, \mathbf{R_3}, \mathbf{C_2}$ and $\mathbf{C_1}$ being discarded)

Now, we again make the set of minimum {4, 5, 7}. From this set minimum entry is 4 which stands in the cell $(\boldsymbol{R_1}, \boldsymbol{C_5})$. **We** select it and discard the row $\boldsymbol{R_1}$ and the column $\boldsymbol{C_5}$. Mark minimum from $\mathbf{C_5}$ - third column.

C / R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10
Column Minimum	-	-	-	5	7

$(\boldsymbol{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{C_2}, \mathbf{C_5}$ and $\mathbf{C_1}$ being discarded)

6) Now the matrix of minimum is as follows

Again the set of minimum is {7,8} Mark the entry $(\boldsymbol{R_4}, \boldsymbol{C_4}) =$ **7** and discard $\boldsymbol{R_4}$ ***and*** $\boldsymbol{C_4}$ so the matrix, now, is as follows

C / R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10
Column Minimum	-	-	-	-	10

7) Now there is only one row $\mathbf{R_5}$ and the last column $\mathbf{C_3}$ in which the minimum entry is 10.

8) This leaves only the last column - $\mathbf{C_3}$ entry. This is $(\mathbf{R_5}, \mathbf{C_3})$ call with entry **is 10.** mark it**.**

C / R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10

9) Basic Feasible Solution Matrix

C / R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10

Resource	**Column**	**Value**

R_1	C_5	4
R_2	C_1	3
R_3	C_2	1
R_4	C_4	7
R_5	C_3	10
Total Sum (BFS)		25

10) Optimality Check

We proceed as follows to check optimality and fix the minimum for each entry.

C R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10

In $(\mathbf{R_5}, \mathbf{C_3})$ cell - the value is 10.
In this row minimum entry $(\mathbf{R_5}, \mathbf{C_5})$ is 5. $|\mathbf{10-5}| = \mathbf{5} = \boldsymbol{L_1}$
In C_5 column there is an allocation in the cell $(\mathbf{R_1}, \mathbf{C_5}) = \mathbf{4}$
This, when shifted to the cell $(\mathbf{R_1}, \mathbf{C_3})$ having value is 7 then $|\mathbf{7-4}| = \mathbf{3} = \boldsymbol{L_2}$ $\boldsymbol{L_2 < L_1}$ it can be interchanged as the above rule because $\boldsymbol{L_2 < L_1}$

The new matrix of allocation is

C R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10

11) New BFS

This modified solution can be written as follows

Resources	Column	Value
R_1	C_3	7
R_2	C_1	3
R_3	C_2	1
R_4	C_4	7
R_5	C_5	5
Total Sum (BFS)		23

12) Again we check the cell value 7 and search for the lower value 5 in that row which is in the column $\mathbf{C_4}$ that is we have $(\mathbf{R_1},\mathbf{C_4}) = 5$. Now we have $|\mathbf{7-5}| = \mathbf{2} = \boldsymbol{L_2}$
In that column C_4 we have an allocated value 7 - cell $(\boldsymbol{R_4},\boldsymbol{C_4})$ and correspondingly in the cell $(\boldsymbol{R_4},\boldsymbol{C_3})$ the entry is 8 so the cell value $(\boldsymbol{R_4},\boldsymbol{C_3})$ =8 and $|\mathbf{8-7}| = \mathbf{1} = \boldsymbol{L_2}$ As $\boldsymbol{L_1}$ is greater than $\boldsymbol{L_2}$ this can interchanged Here $\mathbf{1 < 2}$ so the allocated cell value $(\boldsymbol{R_1},\boldsymbol{C_3})$ can be interchanged with cell $(\mathbf{R_1},\mathbf{C_4}) = 5$ also the cell $(\mathbf{R_4},\mathbf{C_4}) = \mathbf{7}$ can be interchanged with the cell $(\boldsymbol{R_4},\boldsymbol{C_3})$ having the value is 8.

C R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10

In this way we complete optimality check and finally after some iteration reach on the optimal solution found as follows:

C R	C_2	C_1	C_5	C_4	C_3
R_1	8	5	4	5	7
R_2	2	3	9	6	7
R_3	1	8	4	7	5
R_4	12	10	6	7	8
R_5	12	9	5	8	10

From the above matrix, on completion of optimality check, the optimum basic feasible solution can be written as follows

Assignment matrix

Row	Column	Matrix Value
R_1	C_4	5
R_2	C_1	3
R_3	C_2	1
R_4	C_3	8
R_5	C_5	5
Total Sum (BFS)		22

Now at this stage there cannot be further compression as all entries have reached (can be verified) to its minimum.

11. Conclusion

This new method of fixing optimal allocation has proved faster and effective in solving assignment matrix. This has been verified in number of cases and still remains open for discussion. This also indicates the case of multiple optimal solution if it exists.
Maximization problems can also be solved by this method on finding opportunity loss matrix.

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