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Ajay Siwach
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Vinod Bhatia
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Amit Sehgal
Department of Mathematics, Pt. N.R.S Govt. College, Rohtak, Haryana, India

Pankaj Rana
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Corresponding Author:
Ajay Siwach
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Laplacian polynomial of square power graph of dihedral group of order $2n$ with even natural number n

Ajay Siwach, Vinod Bhatia, Amit Sehgal and Pankaj Rana

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Abstract
Square power graph of the dihedral group D_n of order $2n$, $\Gamma_{sq}(D_n)$ is a simple undirected finite graph with vertex set D_n having pairs of different vertices u, v adjacent iff $uv = w^2$ or $vu = w^2$ for any $w \in D_n$ with $w^2 \neq e$ where e is the identity element of D_n. In this research work we have calculated laplacian polynomial of $\Gamma_{sq}(D_n)$ when n is even natural number.

Keywords: Square power graph, dihedral group, degree of a vertex, laplacian polynomial

1. Introduction

Graph Theory is the branch of mathematics that investigates networks and graphs. It originated from the necessity to evaluate a wide range of network-like structures, including the internet, chemicals, road networks, social networks, educational networks, and electrical networks. Spectral graph theory (a subfield of graph theory) studies the relationship between a matrix's eigenvalues and the graph's corresponding structure. The first practical requirement for researching graph eigenvalues was in quantum chemistry in the 1930s, 1940s, and 1950s, notably to define the Hückel molecular orbital theory for unsaturated conjugated hydrocarbons. Several types of graph matrices (adjacency matrix, Laplacian matrix, signless Laplacian matrix, distance matrix, etc.) are widely used in spectral graph theory.

Various structural properties and matrices associated with graphs of groups are studied in [1-5]. In [6-8] various structural properties of the square power graph of the finite Abelian group and its complement graph are studied. Whereas the cubic power graph of finite Abelian group and dihedral group is studied in [9, 10]. k^{th} Power graph of finite abelian group is introduced and degree of vertex is calculated in [11].

Square power graph of the dihedral group D_n of order $2n$, $\Gamma_{sq}(D_n)$ is a simple undirected finite graph with vertex set D_n having pairs of different vertices u, v adjacent iff $uv = w^2$ or $vu = w^2$ for any $w \in D_n$ with $w^2 \neq e$ where e is the identity element of D_n. Laplacian matrix Laplacian matrix, $L(\Gamma_{sq}(D_n))$ is the difference of vertex degree diagonal matrix and adjacency matrix of $\Gamma_{sq}(D_n)$. The characteristic polynomial of $L(\Gamma_{sq}(D_n))$ is known as laplacian polynomial denoted by $\ominus(\Gamma_{sq}(D_n), x)$. If graph Γ is disjoint union of $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ then $\ominus(\Gamma, x) = \prod_{i=1}^{k} \ominus(\Gamma_i, x)$ [12].

2. Laplacian Polynomial

Theorem 2.1 *Let* $\Gamma_{sq}(D_n)$ be square power graph of D_n, dihedral group of order $2n$ where $n = 2m$ is even number then

$$\Gamma_{sq}(G) = \begin{cases} \overline{[2K_1 \cup (\frac{n-4}{4})K_2]} \cup \overline{[\frac{n}{4}K_2]} \cup 2[K_{\frac{n}{2}}] \text{ if } m \text{ is even number,} \\ 2[\overline{K_1 \cup (\frac{n-2}{4})K_2}] \cup 2[K_{\frac{n}{2}}] \text{ if } m \text{ is odd number.} \end{cases}$$

Theorem 2.2 Let $\Gamma_{sq}(D_n)$ be square power graph of dihedral group D_n of order 2n, where n is even number and $u \in D_n$ then

$$\text{Vertex degree, } deg(u) = \begin{cases} \frac{n}{2} - 1 \text{ if } x \in V \cup \{e, x^{\frac{n}{2}}\}, \\ \frac{n}{2} - 2 \text{ if } x \in U \backslash \{e, x^{\frac{n}{2}}\}. \end{cases}$$

Theorem 2.3 Let D_n be dihedral group of order 2n and $n = 2m$ then laplacian polynomial of $\Gamma_{sq}(D_n)$, $\ominus(\Gamma_{sq}(D_n), x) = x^4(\frac{n}{2} - 2 - x)^{\frac{n-2}{2}}(\frac{n}{2} - x)^{\frac{3n-6}{2}}$.

Proof. Let I and O be $\frac{n}{2} \times \frac{n}{2}$ identity and zero matrix respectively.
Case 1. When m is even number
Using Theorem 2.1 and 2.2 we get the Laplacian matrix,

$$L = \begin{bmatrix} L_1 & O & O & O \\ O & L_1 & O & O \\ O & O & L_2 & O \\ O & O & O & L_3 \end{bmatrix}$$

where O is $\frac{n}{2} \times \frac{n}{2}$ zero-matrix,

$$L_1 = \begin{bmatrix} \frac{n}{2} - 1 & -1 & \cdots & -1 \\ -1 & \frac{n}{2} - 1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2} - 1 \end{bmatrix}_{\frac{n}{2} \times \frac{n}{2}},$$

$$L_2 = \begin{bmatrix} \frac{n}{2} - 2 & 0 & \cdots & -1 & -1 \\ 0 & \frac{n}{2} - 2 & \cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2} - 2 & 0 \\ -1 & -1 & \cdots & 0 & \frac{n}{2} - 2 \end{bmatrix}_{\frac{n}{2} \times \frac{n}{2}} \text{ and}$$

$$L_3 = \begin{bmatrix} \frac{n}{2} - 1 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \frac{n}{2} - 1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \frac{n}{2} - 2 & 0 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \frac{n}{2} - 2 & \cdots & -1 & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \frac{n}{2} - 2 & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \frac{n}{2} - 2 \end{bmatrix}_{\frac{n}{2} \times \frac{n}{2}}$$

Thus we have

$$\ominus(\Gamma_{sq}(D_n), x) = \ominus(\overline{[2K_1 \cup (\tfrac{n-4}{4})K_2]}, x) \times \ominus(\overline{[\tfrac{n}{4} K_2]}, x) \times \ominus(K_{\frac{n}{2}}, x) \times \ominus(K_{\frac{n}{2}}, x).$$

$$\ominus(K_{\frac{n}{2}}, x) = |L_1 - xI|$$

$$|L_1 - xI| = \begin{vmatrix} \frac{n}{2} - 1 - x & -1 & \cdots & -1 \\ -1 & \frac{n}{2} - 1 - x & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2} - 1 - x \end{vmatrix},$$

Applying $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{n}{2}}$

$$|L_1 - xI| = -x\begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1 & \frac{n}{2}-1-x & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i + R_1$ for all $i = 2,3,\cdots,\frac{n}{2}$

We get, $|L_1 - xI| = -x\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & \frac{n}{2}-x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{n}{2}-x \end{vmatrix}$

Thus, $|L_1 - xI| = -x(\frac{n}{2} - x)^{\frac{n}{2}-1}$.

$\ominus\,(\overline{[\frac{n}{4}K_2]}, x) = |L_2 - xI|$.

$$|L_2 - xI| = \begin{vmatrix} \frac{n}{2}-2-x & 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & \frac{n}{2}-2-x & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \frac{n}{2}-2-x & 0 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \frac{n}{2}-2-x & \cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots & \frac{n}{2}-2-x & 0 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \frac{n}{2}-2-x \end{vmatrix}$$

Applying, $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{n}{2}}$

$$|L_2 - xI| = -x\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & \frac{n}{2}-2-x & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \frac{n}{2}-2-x & 0 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \frac{n}{2}-2-x & \cdots & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots & \frac{n}{2}-2-x & 0 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \frac{n}{2}-2-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i + R_1$ for all $i = 2,3,\cdots,\frac{n}{2}$

$$|L_2 - xI| = -x\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \frac{n}{2}-1-x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{n}{2}-1-x & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{n}{2}-1-x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i+1}$ for all $i = 3,5,\cdots,\frac{n}{2}-1$

$$|L_2 - xI| = -x \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \frac{n}{2}-1-x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{n}{2}-2-x & -(\frac{n}{2}-2-x) & \cdots & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{n}{2}-2-x & -(\frac{n}{2}-2-x) \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

$$= -x(\frac{n}{2}-2-x)^{\frac{n}{4}-1} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \frac{n}{2}-1-x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i-1}$ for all $i = 4,6,\cdots,\frac{n}{2}$

$$|L_2 - xI| = -x(\frac{n}{2}-2-x)^{\frac{n}{4}-1} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \frac{n}{2}-1-x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{n}{2}-x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{n}{2}-x \end{vmatrix}$$

$$= -x(\frac{n}{2}-2-x)^{\frac{n}{4}-1}(\frac{n}{2}-x)^{\frac{n}{4}-1} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \frac{n}{2}-1-x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

Applying, $R_2 \Rightarrow R_2 - R_1$

$$|L_2 - xI| = -x(\frac{n}{2}-2-x)^{\frac{n}{4}-1}(\frac{n}{2}-x)^{\frac{n}{4}-1}$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & \frac{n}{2}-2-x & -1 & -1 & \cdots & -1 & -1 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

Thus $|L_2 - xI| = -x(\frac{n}{2}-2-x)^{\frac{n}{4}}(\frac{n}{2}-x)^{\frac{n}{4}-1}$.

$\ominus\,(\overline{[2K_1 \cup (\frac{n-4}{4})K_2]}, x) = |L_3 - xI|$.

$$|L_3 - xI| = \begin{vmatrix} \frac{n}{2}-1-x & -1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & \frac{n}{2}-1-x & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \frac{n}{2}-2-x & 0 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \frac{n}{2}-2-x & \cdots & -1 & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \frac{n}{2}-2-x & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \frac{n}{2}-2-x \end{vmatrix}$$

Applying $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{n}{2}}$

$$|L_3 - xI| = -x \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ -1 & \frac{n}{2}-1-x & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \frac{n}{2}-2-x & 0 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \frac{n}{2}-2-x & \cdots & -1 & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \frac{n}{2}-2-x & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \frac{n}{2}-2-x \end{vmatrix}$$

Applying $R_i \Rightarrow R_i + R_1$ for all $i = 2,3,4,\cdots,\frac{n}{2}$

$$|L_3 - xI| = -x \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & \frac{n}{2}-x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{n}{2}-1-x & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{n}{2}-1-x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i+1}$ for all $i = 3,5,\cdots,\frac{n}{2}-1$

$$|L_3 - xI| = -x \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & \frac{n}{2}-x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{n}{2}-2-x & -(\frac{n}{2}-2-x) & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{n}{2}-2-x & -(\frac{n}{2}-2-x) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

$$= -x(\frac{n}{2}-2-x)^{\frac{n}{4}-1} \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & \frac{n}{2}-x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i-1}$ for all $i = 4,6,\cdots,\frac{n}{2}$

$$|L_3 - xI| = -x(\frac{n}{2} - 2 - x)^{\frac{n}{4}-1}\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & \frac{n}{2}-x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{n}{2}-x & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{n}{2}-x \end{vmatrix}$$

Thus $|L_3 - xI| = -x(\frac{n}{2} - 2 - x)^{\frac{n}{4}-1}(\frac{n}{2} - x)^{\frac{n}{4}}$.

Hence$\ominus(\Gamma_{sq}(D_n), x) = \{-x(\frac{n}{2} - x)^{\frac{n}{2}-1}\} \times \{-x(\frac{n}{2} - x)^{\frac{n}{2}-1}\} \times \{-x(\frac{n}{2} - 2 - x)^{\frac{n}{4}}(\frac{n}{2} - x)^{\frac{n}{4}-1}\} \times \{-x(\frac{n}{2} - 2 - x)^{\frac{n}{4}-1}(\frac{n}{2} - x)^{\frac{n}{4}}\} = x^4(\frac{n}{2} - 2 - x)^{\frac{n-2}{2}}(\frac{n}{2} - x)^{\frac{3n-6}{2}}$.

Case 2: When m is odd number

Using Theorem 2.1 and 2.2 we get the Laplacian matrix,

$$L = \begin{bmatrix} L_1 & O & O & O \\ O & L_1 & O & O \\ O & O & L_2 & O \\ O & O & O & L_2 \end{bmatrix}$$

where, $L_1 = \begin{bmatrix} \frac{n}{2}-1 & -1 & \cdots & -1 \\ -1 & \frac{n}{2}-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2}-1 \end{bmatrix}_{\frac{n}{2}\times\frac{n}{2}}$ and

$$L_2 = \begin{bmatrix} \frac{n}{2}-1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \frac{n}{2}-2 & 0 & \cdots & -1 & -1 & -1 \\ -1 & 0 & \frac{n}{2}-2 & \cdots & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & -1 & \frac{n}{2}-2 & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \frac{n}{2}-2 \end{bmatrix}_{\frac{n}{2}\times\frac{n}{2}}$$

Thus we have

$\ominus(\Gamma_{sq}(D_n), x) = \ominus(\overline{[K_1 \cup (\frac{n-2}{4})K_2]}) \times \ominus(\overline{[K_1 \cup (\frac{n-2}{4})K_2]}) \times \ominus(K_{\frac{n}{2}}, x) \times \ominus(K_{\frac{n}{2}}, x)$.

$\ominus(K_{\frac{n}{2}}, x) = |L_1 - xI|$.

$$|L_1 - xI| = \begin{vmatrix} \frac{n}{2}-1-x & -1 & \cdots & -1 \\ -1 & \frac{n}{2}-1-x & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{n}{2}}$

$$|L_1 - xI| = -x\begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1 & \frac{n}{2}-1-x & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i + R_1$ for all $i = 2,3,\cdots,\frac{n}{2}$

$$|L_1 - xI| = -x\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & \frac{n}{2}-x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{n}{2}-x \end{vmatrix}$$

Thus, $|L_1 - xI| = -x(\frac{n}{2} - x)^{\frac{n}{2}-1}$.

$\ominus\left(\overline{[K_1 \cup (\frac{n-2}{4})K_2]}\right) = |L_2 - xI|$.

$$|L_2 - xI| = \begin{vmatrix} \frac{n}{2}-1-x & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & \frac{n}{2}-2-x & 0 & \cdots & -1 & -1 & -1 \\ -1 & 0 & \frac{n}{2}-2-x & \cdots & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & -1 & \frac{n}{2}-2-x & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \frac{n}{2}-2-x \end{vmatrix}$$

Applying, $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{n}{2}}$

$$|L_2 - xI| = -x\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & \frac{n}{2}-2-x & 0 & \cdots & -1 & -1 & -1 \\ -1 & 0 & \frac{n}{2}-2-x & \cdots & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & -1 & \frac{n}{2}-2-x & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \frac{n}{2}-2-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i + R_1$ for all $i = 2,3,\cdots,\frac{n}{2}$

$$|L_2 - xI| = -x\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & \frac{n}{2}-1-x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{n}{2}-1-x & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i+1}$ for all $i = 2,4,\cdots,\frac{n}{2} - 1$

$$|L_2 - xI| = -x\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & \frac{n}{2}-2-x & -(\frac{n}{2}-2-x) & \cdots & 0 & 0 & 0 \\ 0 & 1 & \frac{n}{2}-1-x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \frac{n}{2}-2-x & -(\frac{n}{2}-2-x) \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{n}{2}-1-x \end{vmatrix}$$

$$= -x(\frac{n}{2} - 2 - x)^{\frac{n-2}{4}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \frac{n}{2} - 1 - x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{n}{2} - 1 - x \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i-1}$ for all $i = 3,5,\cdots,\frac{n}{2}$

$$|L_2 - xI| = -x(\frac{n}{2} - 2 - x)^{\frac{n-2}{4}} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{n}{2} - x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{n}{2} - x \end{vmatrix}$$

Thus, $|L_2 - xI| = -x(\frac{n}{2} - 2 - x)^{\frac{n-2}{4}}(\frac{n}{2} - x)^{\frac{n-2}{4}}$.

Hence $\ominus(\Gamma_{sq}(D_n), x) = \{-x(\frac{n}{2} - x)^{\frac{n}{2}-1}\} \times \{-x(\frac{n}{2} - x)^{\frac{n}{2}-1}\} \times \{-x(\frac{n}{2} - 2 - x)^{\frac{n-2}{4}}(\frac{n}{2} - x)^{\frac{n-2}{4}}\} \times \{-x(\frac{n}{2} - 2 - x)^{\frac{n-2}{4}}(\frac{n}{2} - x)^{\frac{n-2}{4}}\} = x^4(\frac{n}{2} - 2 - x)^{\frac{n-2}{2}}(\frac{n}{2} - x)^{\frac{3n-6}{2}}$.

Hence the required result.

3. Conclusion
In this research laplacian polynomial of the square power graph with vertex set dihedral graph of order $2n$ for even natural number n is calculated.

4. References

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