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Ajay Siwach
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Vinod Bhatia
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Amit Sehgal
Department of Mathematics, Pt. NRS Govt. College, Rohtak, Haryana, India

Pankaj Rana
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Corresponding Author:
Ajay Siwach
Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak, Haryana, India

Characteristic polynomial of maximum and minimum matrix of square power graph of dihedral group of order $2n$ with odd natural number n

Ajay Siwach, Vinod Bhatia, Amit Sehgal and Pankaj Rana

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Abstract
For dihedral group D_n of order $2n$ with identity element R_0, Square power graph of dihedral group D_n is an undirected finite, simple graph having pair of distinct vertices X_1, X_2 have edge iff $X_1X_2 = X^2$ or $X_2X_1 = X^2$ for any $X \in D_n$ where $X^2 \neq R_0$. In this research, we have calculated characteristic polynomials of degree of vertex based matrices such as maximum and minimum matrix of the Square power graph of dihedral group D_n of order $2n$ with odd natural number n.

Keywords: Dihedral group, square power graph, vertex degree, characteristic Polynomial, maximum matrix, minimum matrix

Introduction

Recently various graphs having groups as their vertex set are studied. Square power graph of finite abelian groups are studied in [1, 2] whereas cubic power graph of finite abelian group in [3] and for dihedral group in [4]. Degree of vertices of k^{th} power graph of finite abelian group is calculated in [5]. Laplacian polynomial of square power of D_n for even n is calculated in [6] whereas for power graphs in [8]. Various degree-based matrices characteristics polynomials are calculated in [7].

Throughout this paper, we have used $D_n = \{R_0, R_{\frac{360}{n}}, R_{\frac{2\times360}{n}}, R_{\frac{3\times360}{n}}, \cdots R_{\frac{(n-1)\times360}{n}}, F_{a_1}, F_{a_2}, F_{a_3}, \cdots, F_{a_n}\}$, here $R_{\frac{i\times360}{n}}$ are rotation elements for $0 \leq i \leq n-1$ and $R_{360} = R_0$; and F_{a_j} are reflection elements all with order 2 for $1 \leq j \leq n$.

Definition 1.1 [9] The maximum matrix of $\Gamma_{cpg}(D_n)$, denoted by $Max(\Gamma_{cpg}(D_n)) = [max_{ij}]_{2n\times2n}$ whose $(i,j)^{th}$ entry is

$$max_{ij} = \begin{cases} max\left\{deg_{\Gamma_{cpg}(D_n)}(X_i), deg_{\Gamma_{cpg}(D_n)}(X_j)\right\}, \text{if } X_i \neq X_j \text{ and they are adjacent} \\ 0, \text{otherwise.} \end{cases}$$

Definition 1.2 [10] The minimum matrix of $\Gamma_{cpg}(D_n)$, denoted by $Min(\Gamma_{cpg}(D_n)) = [min_{ij}]_{2n\times2n}$ whose $(i,j)^{th}$ entry is

$$min_{ij} = \begin{cases} min\left\{deg_{\Gamma_{cpg}(D_n)}(X_i), deg_{\Gamma_{cpg}(D_n)}(X_j)\right\}, \text{if } X_i \neq X_j \text{ and they are adjacent} \\ 0, \text{otherwise.} \end{cases}$$

Characteristic Polynomial

Theorem 2.1 Let $\Gamma_{sq}(G)$ be square power graph of D_n and n is odd number then degree of any vertex X

$$deg_{\Gamma_{sq}(G)}(X) = \begin{cases} n-1 \text{ if } X = R_0, \\ n-2 \text{ if } X \text{ is any rotation element vertex other than } R_0, \\ n-1 \text{ if } X \text{ is any reflection element vertex.} \end{cases}$$

Theorem 2.2 Let D_n be dihedral group of order $2n = m$ with odd number n and $P_{Max(\Gamma_{sq}(D_n))}$ be characteristic polynomial of the maximum matrix of $\Gamma_{sq}(D_n)$. Then, $P_{Max(\Gamma_{sq}(D_n))} = (-1)^{\frac{m-4}{2}} \lambda^{\frac{m-2}{4}} (\lambda + m - 4)^{\frac{m-6}{4}} \{\lambda^2 - \lambda(\frac{m^2-10m+24}{4}) - (\frac{m^3-6m^2+12m-8}{8})\}\{-\lambda + (\frac{m}{2}-1)^2\}(-\lambda - \frac{m}{2} + 1)^{\frac{m-2}{2}}$.

Proof. Let I and O be $\frac{m}{2} \times \frac{m}{2}$ order identity and zero matrices respectively.
Using Theorem 2.1 we get the maximum matrix,

$$Max(\Gamma_{sq}(D_n)) = \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix}$$

where,
$$M_1 = \begin{bmatrix} 0 & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & 0 & 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-1 & 0 & 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & 0 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & 0 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & 0 \end{bmatrix}_{\frac{m}{2}\times\frac{m}{2}},$$

$$M_2 = \begin{bmatrix} 0 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & 0 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & 0 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & 0 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & 0 \end{bmatrix}_{\frac{m}{2}\times\frac{m}{2}}$$

Thus, $P_{Max(\Gamma_{sq}(D_n))} = |M_1 - \lambda I| \times |M_2 - \lambda I|$.

$$|M_1 - \lambda I| = \begin{vmatrix} -\lambda & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & -\lambda & 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-1 & 0 & -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & -\lambda & 0 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & -\lambda & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & -\lambda & 0 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & -\lambda \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i+1}$ for all $i = 2,4,\cdots,\frac{m}{2}-1$

$$|M_1-\lambda I|=\begin{vmatrix} -\lambda & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & 0 & -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ 0 & 0 & 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & -\lambda & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & -\lambda \end{vmatrix}$$

$$=\lambda^{\frac{m-2}{4}}\begin{vmatrix} -\lambda & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & 0 & -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & -\lambda & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & -\lambda \end{vmatrix}$$

Applying, $C_i \Rightarrow C_i + C_{i-1}$ for all $i = 3,5,\cdots,\frac{m}{2}$

$$|M_1-\lambda I|=(-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}}\begin{vmatrix} -\lambda & \frac{m}{2}-1 & 2(\frac{m}{2}-1) & \frac{m}{2}-1 & 2(\frac{m}{2}-1) & \cdots & \frac{m}{2}-1 & 2(\frac{m}{2}-1) \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & 0 & -\lambda & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & 0 & -\lambda & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{m}{2}-1 & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & 0 & -\lambda \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_3$ for all $i = 5,7,\cdots,\frac{m}{2}$

$$|M_1-\lambda I|=(-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}}$$

$$\times\begin{vmatrix} -\lambda & \frac{m}{2}-1 & 2(\frac{m}{2}-1) & \frac{m}{2}-1 & 2(\frac{m}{2}-1) & \cdots & \frac{m}{2}-1 & 2(\frac{m}{2}-1) \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-1 & 0 & -\lambda & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{m}{2}-2 & 2(\frac{m}{2}-2)+\lambda & -(\frac{m}{2}-2) & -\lambda-2(\frac{m}{2}-2) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & \frac{m}{2}-2 & 2(\frac{m}{2}-2)+\lambda & 0 & 0 & \cdots & -(\frac{m}{2}-2) & -\lambda-2(\frac{m}{2}-2) \end{vmatrix}$$

Extending through $a_{22}, a_{44}, \cdots, a_{(\frac{m}{2}-1)(\frac{m}{2}-1)}$

$$|M_1-\lambda I|=(-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}}\begin{vmatrix} -\lambda & 2(\frac{m}{2}-1) & 2(\frac{m}{2}-1) & \cdots & 2(\frac{m}{2}-1) \\ \frac{m}{2}-1 & -\lambda & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ 0 & 2(\frac{m}{2}-2)+\lambda & -\lambda-2(\frac{m}{2}-2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2(\frac{m}{2}-2)+\lambda & 0 & \cdots & -\lambda-2(\frac{m}{2}-2) \end{vmatrix}$$

$$= (-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}\begin{vmatrix} -\lambda & 2(\frac{m}{2}-1) & 2(\frac{m}{2}-1) & \cdots & 2(\frac{m}{2}-1) \\ \frac{m}{2}-1 & -\lambda & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ 0 & 1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & -1 \end{vmatrix}$$

Applying, $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{m+2}{4}}$

$$|M_1 - \lambda I| = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}$$

$$\begin{vmatrix} -\lambda & 2(\frac{m}{2}-1)(\frac{m-2}{4}) & 2(\frac{m}{2}-1) & \cdots & 2(\frac{m}{2}-1) \\ \frac{m}{2}-1 & -\lambda + 2(\frac{m}{2}-2)(\frac{m-6}{4}) & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$|M_1 - \lambda I| = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}\begin{vmatrix} -\lambda & (\frac{m}{2}-1)(\frac{m}{2}-1) \\ \frac{m}{2}-1 & -\lambda + (\frac{m}{2}-2)(\frac{m}{2}-3) \end{vmatrix}$$

$$|M_1 - \lambda I| = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}\{\lambda^2 - (\frac{m^2-10m+24}{4})\lambda - (\frac{m^3-6m^2+12m-8}{8})\}$$

$$|M_2 - \lambda I| = \begin{vmatrix} -\lambda & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & -\lambda & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & -\lambda & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & -\lambda & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & -\lambda \end{vmatrix}$$

Applying, $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{m}{2}}$

$$|M_2 - \lambda I| = \begin{vmatrix} -\lambda + (\frac{m}{2}-1)^2 & -\lambda + (\frac{m}{2}-1)^2 & -\lambda + (\frac{m}{2}-1)^2 & \cdots & -\lambda + (\frac{m}{2}-1)^2 & -\lambda + (\frac{m}{2}-1)^2 \\ \frac{m}{2}-1 & -\lambda & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & -\lambda & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & -\lambda & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & -\lambda \end{vmatrix}$$

$$= \{-\lambda + (\frac{m}{2}-1)^2\}\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \frac{m}{2}-1 & -\lambda & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & -\lambda & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & -\lambda & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & -\lambda \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - (\frac{m}{2}-1)R_1$ for all $i = 2,3,\cdots,\frac{m}{2}$

$|M_2 - \lambda I| = \{-\lambda + (\frac{m}{2} - 1)^2\}$

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -\lambda - \frac{m}{2} + 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda - \frac{m}{2} + 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\lambda - \frac{m}{2} + 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda - \frac{m}{2} + 1 \end{vmatrix}$$

$|M_2 - \lambda I| = \{-\lambda + (\frac{m}{2} - 1)^2\}(-\lambda - \frac{m}{2} + 1)^{\frac{m}{2}-1}$.

Hence, $P_{Max(\Gamma_{sq}(D_m))} = |M_1 - \lambda I| \times |M_2 - \lambda I| = \{(-1)^{\frac{m-4}{2}} \lambda^{\frac{m-2}{4}} (m - 4 + \lambda)^{\frac{m-6}{4}} \{\lambda^2 - (\frac{m^2-10m+24}{4})\lambda - (\frac{m^3-6m^2+12m-8}{8})\}\} \times \{\{-\lambda + (\frac{m}{2} - 1)^2\}(-\lambda - \frac{m}{2} + 1)^{\frac{m}{2}-1}\} = (-1)^{\frac{m-4}{2}} \lambda^{\frac{m-2}{4}} (\lambda + m - 4)^{\frac{m-6}{4}} \{\lambda^2 - \lambda(\frac{m^2-10m+24}{4}) - (\frac{m^3-6m^2+12m-8}{8})\}\{-\lambda + (\frac{m}{2} - 1)^2\}(-\lambda - \frac{m}{2} + 1)^{\frac{m-2}{2}}$.

Theorem 2.3 *Let D_n be dihedral group of order $2n = m$ with odd number n and $P_{Min(\Gamma_{sq}(D_n))}$ be characteristic polynomial of the minimum matrix of $\Gamma_{sq}(D_n)$. Then,* $P_{Min(\Gamma_{sq}(D_n))} = (-1)^{\frac{m-4}{2}} \lambda^{\frac{m-2}{4}} (m - 4 + \lambda)^{\frac{m-6}{4}} \{\lambda^2 - (\frac{m^2-10m+24}{4})\lambda - (\frac{m^3-10m^2+32m-32}{8})\}\{-\lambda + (\frac{m}{2} - 1)^2\}(-\lambda - \frac{m}{2} + 1)^{\frac{m-2}{2}}$.

Proof. Let I and O be $\frac{m}{2} \times \frac{m}{2}$ order identity and zero matrices respectively.
Using Theorem 2.1 we get the minimum matrix,

$$Min(\Gamma_{sq}(D_n)) = \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix}$$

Where, $M_1 = \begin{bmatrix} 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & 0 & 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & 0 & 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & 0 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & 0 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & 0 \end{bmatrix}_{\frac{m}{2} \times \frac{m}{2}}$,

$$M_2 = \begin{bmatrix} 0 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & 0 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & 0 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & 0 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & 0 \end{bmatrix}_{\frac{m}{2} \times \frac{m}{2}}$$

Thus, $P_{Min(\Gamma_{sq}(D_n))} = |M_1 - \lambda I| \times |M_2 - \lambda I|$.

$$|M_1-\lambda I|=\begin{vmatrix} -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & -\lambda & 0 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & 0 & -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & -\lambda & 0 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & -\lambda & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & -\lambda & 0 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & -\lambda \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_{i+1}$ for all $i = 2,4,\cdots,\frac{m}{2}-1$

$$|M_1-\lambda I|=\begin{vmatrix} -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & 0 & -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ 0 & 0 & 0 & -\lambda & \lambda & \cdots & 0 & 0 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & -\lambda & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -\lambda & \lambda \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & -\lambda \end{vmatrix}$$

$$=\lambda^{\frac{m-2}{4}}\begin{vmatrix} -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & 0 & -\lambda & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & 0 & -\lambda & \cdots & \frac{m}{2}-2 & \frac{m}{2}-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \frac{m}{2}-2 & \cdots & 0 & -\lambda \end{vmatrix}$$

Applying, $C_i \Rightarrow C_i + C_{i-1}$ for all $i = 3,5,\cdots,\frac{m}{2}$

$$|M_1-\lambda I|=(-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}}\begin{vmatrix} -\lambda & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & 0 & -\lambda & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & 0 & -\lambda & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{m}{2}-2 & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & 0 & -\lambda \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - R_3$ for all $i = 5,7,\cdots,\frac{m}{2}$

$$|M_1-\lambda I| = (-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}} \times \begin{vmatrix} -\lambda & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{m}{2}-2 & 0 & -\lambda & \frac{m}{2}-2 & 2(\frac{m}{2}-2) & \cdots & \frac{m}{2}-2 & 2(\frac{m}{2}-2) \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{m}{2}-2 & 2(\frac{m}{2}-2)+\lambda & -(\frac{m}{2}-2) & -\lambda-2(\frac{m}{2}-2) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & \frac{m}{2}-2 & 2(\frac{m}{2}-2)+\lambda & 0 & 0 & \cdots & -(\frac{m}{2}-2) & -\lambda-2(\frac{m}{2}-2) \end{vmatrix}$$

Extending through $a_{22}, a_{44}, \cdots, a_{(\frac{m}{2}-1)(\frac{m}{2}-1)}$

$$|M_1-\lambda I| = (-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}} \begin{vmatrix} -\lambda & 2(\frac{m}{2}-2) & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ \frac{m}{2}-2 & -\lambda & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ 0 & 2(\frac{m}{2}-2)+\lambda & -\lambda-2(\frac{m}{2}-2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2(\frac{m}{2}-2)+\lambda & 0 & \cdots & -\lambda-2(\frac{m}{2}-2) \end{vmatrix}$$

$$= (-1)^{\frac{m-2}{4}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}} \begin{vmatrix} -\lambda & 2(\frac{m}{2}-2) & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ \frac{m}{2}-2 & -\lambda & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ 0 & 1 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & -1 \end{vmatrix}$$

Applying, $C_2 \Rightarrow C_2 + C_3 + \cdots + C_{\frac{m+2}{4}}$

$$|M_1-\lambda I| = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}$$

$$\begin{vmatrix} -\lambda & 2(\frac{m}{2}-2)(\frac{m-2}{4}) & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ \frac{m}{2}-2 & -\lambda+2(\frac{m}{2}-2)(\frac{m-6}{4}) & 2(\frac{m}{2}-2) & \cdots & 2(\frac{m}{2}-2) \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$|M_1-\lambda I| = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}} \begin{vmatrix} -\lambda & (\frac{m}{2}-1)(\frac{m}{2}-2) \\ \frac{m}{2}-2 & -\lambda+(\frac{m}{2}-2)(\frac{m}{2}-3) \end{vmatrix}$$

$$|M_1-\lambda I| = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}\{\lambda^2-(\frac{m^2-10m+24}{4})\lambda-(\frac{m^3-10m^2+32m-32}{8})\}$$

$$|M_2-\lambda I| = \begin{vmatrix} -\lambda & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & -\lambda & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & -\lambda & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & -\lambda & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & -\lambda \end{vmatrix}$$

Applying, $R_1 \Rightarrow R_1 + R_2 + \cdots + R_{\frac{m}{2}}$

$$|M_2-\lambda I| = \begin{vmatrix} -\lambda+(\frac{m}{2}-1)^2 & -\lambda+(\frac{m}{2}-1)^2 & -\lambda+(\frac{m}{2}-1)^2 & \cdots & -\lambda+(\frac{m}{2}-1)^2 & -\lambda+(\frac{m}{2}-1)^2 \\ \frac{m}{2}-1 & -\lambda & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & -\lambda & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & -\lambda & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & -\lambda \end{vmatrix}$$

$$= \{-\lambda+(\frac{m}{2}-1)^2\} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \frac{m}{2}-1 & -\lambda & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & -\lambda & \cdots & \frac{m}{2}-1 & \frac{m}{2}-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & -\lambda & \frac{m}{2}-1 \\ \frac{m}{2}-1 & \frac{m}{2}-1 & \frac{m}{2}-1 & \cdots & \frac{m}{2}-1 & -\lambda \end{vmatrix}$$

Applying, $R_i \Rightarrow R_i - (\frac{m}{2}-1)R_1$ for all $i = 2,3,\cdots,\frac{m}{2}$

$$|M_2-\lambda I| = \{-\lambda+(\frac{m}{2}-1)^2\} \times$$

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -\lambda-\frac{m}{2}+1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda-\frac{m}{2}+1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\lambda-\frac{m}{2}+1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda-\frac{m}{2}+1 \end{vmatrix}$$

$$|M_2-\lambda I| = \{-\lambda+(\frac{m}{2}-1)^2\}(-\lambda-\frac{m}{2}+1)^{\frac{m}{2}-1}.$$

Hence $P_{Min(\Gamma_{sq}(D_n))} = |M_1-\lambda I| \times |M_2-\lambda I| = \{(-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}\{\lambda^2-(\frac{m^2-10m+24}{4})\lambda-(\frac{m^3-10m^2+32m-32}{8})\}\} \times \{\{-\lambda+(\frac{m}{2}-1)^2\}(-\lambda-\frac{m}{2}+1)^{\frac{m}{2}-1}\} = (-1)^{\frac{m-4}{2}}\lambda^{\frac{m-2}{4}}(m-4+\lambda)^{\frac{m-6}{4}}\{\lambda^2-(\frac{m^2-10m+24}{4})\lambda-(\frac{m^3-10m^2+32m-32}{8})\}\{-\lambda+(\frac{m}{2}-1)^2\}(-\lambda-\frac{m}{2}+1)^{\frac{m-2}{2}}$.

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