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Studying asymptotic behavior of bi-shadowing properties in multiple product dynamical systems in group space

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Abstract
In this paper, we presented a set of basic concepts in the group space, such as the definition of the bi-shadowing in its parameterized and unparameterized types. In the section three, we presented a set of theorems to study the bi-shadowing property of two dynamic systems, one of which is bi-shadowing and the other is (α, β)-bi-shadowing. In the section four, we studied the bi-shadowing property for a group space of dynamic systems that achieve the bi-shadowing property in two types mentioned.

Keywords: Bi-shadowing, group space, dynamical systems, product systems, multiple product systems

Introduction

Shadowing theory is a major theory in dynamical systems. It has an important role in studying asymptotic behavior and stability in systems, see also [1, 2]. With it, computer calculations of the dynamic system can be confirmed and the existence of the true orbit of the system near the pseudo-orbit, see [3, 4].

The first to develop the concept of shadowing was Walters, P. [5], see more [6, 7, 8] and the concept of inverse shadowing and the concept of bi-shadowing were developed by other researchers, see [9, 10, 11]. Ajam, M. H. O. in [12] and Al-Badarneh, A. A. in [13] studied the property of the bi-shadowing in the multiple dynamical system.

In recent years, the dynamical systems through which these properties are studied have been expanded to include those dynamical systems in group space. See [14, 15, 16, 17, 18, 19].

In the section two of this paper, we will present a set of basic concepts in our paper, such as the definition of the (α, β)-bi-shadowing and the bi-shadowing, which we need in theorems and their proofs.

In the section three, we will present a set of theorems to study the relationship between two dynamic systems, one of which is (α, β)-bi-shadowing and the other is bi-shadowing with product systems.

In the section four, we will study the relationship between a dynamic systems that achieve these two concepts with multiple product systems.

Preliminaries

In the definitions, theories and results that will be mentioned in this paper we will suppose that the action $\Phi: \mathbb{G} \times \chi \to \chi$ of a finitely generated group $\mathbb{G}$ with respect to the generating set $\mathbb{S}$, ($\mathbb{S}$ is any generating set of $\mathbb{G}$) on a metric group space $(\chi, \mathbb{d}_\chi)$.

We need to introduce the true-orbit and pseudo-orbit definitions.

Definition 2.1: [18] We called the sequence $\{x_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ is true-orbit of the action Φ if.

$\Phi(\mathbb{s}, x_{\mathbb{g}}) = x_{\mathbb{g}\mathbb{s}}$ for $\mathbb{s} \in \mathbb{S}, \mathbb{g} \in \mathbb{G}$.

For $\rho > 0$, we called the sequence $\{y_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ is ρ-pseudo-orbit of an action Φ if.

$\mathbb{d}_\chi(\Phi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{gs}}) < \rho$, for $\mathbb{s} \in \mathbb{S}, \mathbb{g} \in \mathbb{G}$.

Below we mention the bi-shadowing definition, therefore, we need to provide the definition of distance between two actions. For Φ and Ψ are actions, the distance between them is given by.

$$\mathbb{d}_{\chi_0}(\Phi, \Psi) = \sup_{x\in\chi}\{\mathbb{d}_\chi(\Phi(\mathbb{s}, x), \Psi(\mathbb{s}, x))\} \text{ for } \mathbb{s} \in \mathbb{S}.$$

Definition 2.2: [17] Let $(\chi, \mathbb{d}_\chi)$ be metric group space, and $\Phi: \mathbb{G} \times \chi \to \chi$ be an action. The action Φ is called bi-shadowing with positive parameters α and β if there exists $0 < \delta \le \beta$ such that for any δ-pseudo-orbit $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of Φ and any action $\mathbb{f}: \mathbb{G} \times \chi \to \chi$ satisfying.

$$\mathbb{d}_{\chi_0}(\Phi, \mathbb{f}) \le \beta - \delta,$$

Then there exists a true-orbit $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{f}$ such that.

$$\mathbb{d}_\chi(x_{\mathbb{g}}, y_{\mathbb{g}}) \le \alpha\left(\delta + \mathbb{d}_{\chi_0}(\Phi, \Psi)\right) \le \alpha\beta, \text{ for all } \mathbb{g} \in \mathbb{G}.$$

Definition 2.3: [19] Let $(\chi, \mathbb{d}_\chi)$ be metric group space, and $\Phi: \mathbb{G} \times \chi \to \chi$ be an action. The action Φ is called bi-shadowing if for all $\beta > 0$ there exists $\alpha > 0$ and $0 < \delta \le \beta/\alpha$ such that for any δ-pseudo orbit $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of Φ and any action $\mathbb{f}: \mathbb{G} \times \chi \to \chi$ satisfying.

$$\mathbb{d}_{\chi_0}(\Phi, \mathbb{f}) \le \beta - \delta,$$

then there exists a true-orbit $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{f}$ such that

$$\mathbb{d}_\chi(x_{\mathbb{g}}, y_{\mathbb{g}}) \le \alpha\left(\delta + \mathbb{d}_{\chi_0}(\Phi, \Psi)\right) \le \beta, \text{ for all } \mathbb{g} \in \mathbb{G}.$$

Definition 2.4: [13] We assume that we have two compact metric group spaces $(\chi, \mathbb{d}_\chi)$ and $(\Upsilon, \mathbb{d}_\Upsilon)$ then their product metric group space $\chi \times \Upsilon$ with metric defined by

$$\mathbb{D}((x_1, y_1), (x_2, y_2)) = \max\{\mathbb{d}_\chi(x_1, x_2), \mathbb{d}_\Upsilon(y_1, y_2)\}$$

for $(x_1, y_1), (x_2, y_2) \in \chi \times \Upsilon$. We also consider actions $\Phi: \mathbb{G} \times \chi \to \chi$ and $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$, and the product $\Phi \times \Psi: \mathbb{G} \times \chi \times \Upsilon \to \chi \times \Upsilon$, where the product action is defined by $(\Phi \times \Psi)(\mathbb{s}, (x, y)) = (\Phi(\mathbb{s}, x), \Psi(\mathbb{s}, y))$ for $\mathbb{s} \in \mathbb{S}$.

Product System

The proof of the following lemma is straightforward, so will be omitted.

Lemma 3.1: Let $\Phi: \mathbb{G} \times \chi \to \chi$, $\mathbb{f}: \mathbb{G} \times \chi \to \chi$, $\Psi: \mathbb{G} \times \Upsilon \to \Upsilon$ and $\mathbb{h}: \mathbb{G} \times \Upsilon \to \Upsilon$ be actions and let $a = \sup_{x\in\chi}\{\mathbb{d}_\chi(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\}$ and $b = \sup_{y\in\Upsilon}\{\mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then we have.

$$\sup_{x\in\chi, y\in\Upsilon}\left\{\max\{\mathbb{d}_\chi(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)), \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right\} = \max\{a, b\} \text{ for } \mathbb{s} \in \mathbb{S}.$$

Theorem 3.2. Let both Φ and Ψ be (α, β)-bi-shadowing. Then the product system $\Phi \times \Psi$ is (α, β)-bi-shadowing.

Proof. Let $\{(x_{\mathbb{g}}, y_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ be a δ-pseudo orbit of an action $\Phi \times \Psi$, with $0 \le \delta \le \beta$, and let $\Phi \times \Psi: \mathbb{G} \times \chi \times \Upsilon \to \chi \times \Upsilon$ be an action and satisfying.

$$\mathbb{d}_{\chi\times\Upsilon_0}(\Phi, \Psi) = \sup_{(x,y)\in\chi\times\Upsilon}\left\{\mathbb{D}\left((\Phi \times \Psi)(\mathbb{s}, (x, y)), (\mathbb{f} \times \mathbb{h})(\mathbb{s}, (x, y))\right)\right\} \le \beta - \delta,$$

for $\mathbb{s} \in \mathbb{S}$. (1)

Since $\{(x_{\mathbb{g}}, y_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ is a δ-pseudo orbit of an action $\Phi \times \Psi$ we have, for each $\mathbb{g} \in \mathbb{G}$, that

$$\max\{\mathbb{d}_\chi(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{gs}}), \mathbb{d}_\Upsilon(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{gs}})\} = \mathbb{D}\left(\left(\Phi(\mathbb{s}, x_{\mathbb{g}}), \Psi(\mathbb{s}, y_{\mathbb{g}})\right), (x_{\mathbb{gs}}, y_{\mathbb{gs}})\right) = \mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (x_{\mathbb{gs}}, y_{\mathbb{gs}})\right) < \delta, \mathbb{s} \in \mathbb{S}.$$

Thus, $\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}}) < \delta$ and $\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{g}\mathbb{s}}) < \delta$, for $\mathbb{g} \in \mathbb{G}$, which implies that both $\{x_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ of Φ and $\{y_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ of Ψ are δ-pseudo trajectories. Since both Φ and Ψ are (α, β)-bi-shadowing, then for any actions $\mathbb{f}: \mathbb{G} \times \chi \to \chi$ and $\mathbb{h}: \mathbb{G} \times \Upsilon \to \Upsilon$ satisfying

$$\mathbb{d}_{\chi_0}(\Phi, \mathbb{f}) = \sup_{x \in \chi}\{\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\} \leq \beta - \delta \text{ for } \mathbb{s} \in \mathbb{S}, \tag{2}$$

and

$$\mathbb{d}_{\Upsilon_0}(\Psi, \mathbb{h}) = \sup_{y \in \Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\} \leq \beta - \delta \text{ for } \mathbb{s} \in \mathbb{S}, \tag{3}$$

there exist true trajectories $\{w_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{f}$ and $\{z_{\mathbb{g}} : \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{h}$ such that

$$\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) \leq \alpha\left(\delta + \mathbb{d}_{\chi_0}(\Phi, \mathbb{f})\right), \text{ for } \mathbb{g} \in \mathbb{G} \tag{4}$$

and

$$\mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}}) \leq \alpha\left(\delta + \mathbb{d}_{\Upsilon_0}(\Psi, \mathbb{h})\right), \text{ for } \mathbb{g} \in \mathbb{G} \tag{5}$$

We consider the following the following Lemma.

Lemma 3.3: Assume that $\mathbb{d}_{\chi_0}(\Phi, \mathbb{f}) > \mathbb{d}_{\Upsilon_0}(\Psi, \mathbb{h})$ with the relation (4) and (5) then.

$$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right)$$

$$\leq \alpha\left(\delta + \sup_{(x,y) \in \chi \times \Upsilon} \mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right), \mathbb{g} \in \mathbb{G}.$$

Proof

Case 1: For the values of $\mathbb{g} \in \mathbb{G}$, for which $\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) > \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})$, and using the relation (4), we have.

$$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right) = max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\} = \mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) \leq \alpha\left(\delta + \sup_{x \in \chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\right)$$

So, for every $\in \chi, y \in \Upsilon$, and using Lemma 3.1, we have.

$$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right) \leq \alpha\left(\delta + \sup_{x \in \chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\right)$$

$$= \alpha\left(\delta + max\left\{\sup_{x \in \chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)), \sup_{y \in \Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right\}\right) = \alpha\left(\delta + \sup_{x \in \chi, y \in \Upsilon} max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\}\right)$$

$$= \alpha\left(\delta + \sup_{(x,y) \in \chi \times \Upsilon} \mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right)$$

Case 2: For the values of $\mathbb{g} \in \mathbb{G}$, for which $\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) < \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})$, and using the relation (5), we have.

$$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right) = max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\} = \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})$$

$$\leq \alpha\left(\delta + \sup_{y \in \Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right) \leq \alpha\left(\delta + \sup_{x \in \chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\right)$$

From the argument of Case 1 above we obtain.

$$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right) \leq \alpha\left(\delta + \sup_{(x,y) \in \chi \times \Upsilon} \mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right)$$

Case 3: For the values of $\mathbb{g} \in \mathbb{G}$, for which $\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) = \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})$, we have the same result as in Case 1 and Case 2.
By combining the three cases, we have.

$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right)$

$\leq \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right)$ for $\mathbb{g}\in\mathbb{G}$.

This ends the proof of Lemma 3.3.

Now we complete the proof of Theorem 3.2
We may assume $\mathbb{d}_{\chi_0}(\Phi,\mathbb{f}) > \mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h})$, as the other direction can be treated similarly. Then By Lemma 3.3 we get

$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right)$

$\leq \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right)$ for $\mathbb{g}\in\mathbb{G}$.

Finally, the sequence $\{(w_{\mathbb{g}}, z_{\mathbb{g}}): \mathbb{g}\in\mathbb{G}\}$ is a true orbit of $\mathbb{f}\times\mathbb{h}$ since

$(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(w_{\mathbb{g}}, z_{\mathbb{g}})\right) = \left(\mathbb{f}(\mathbb{s}, w_{\mathbb{g}}), \mathbb{h}(\mathbb{s}, z_{\mathbb{g}})\right) = (w_{\mathbb{g}\mathbb{s}}, z_{\mathbb{g}\mathbb{s}}), \mathbb{g}\in\mathbb{G}$

This means that $\Phi\times\Psi$ is (α,β)-bi-shadowing.
This ends the proof of Theorem 3.2.

The theorem below is similar to Theorem 3.2, but we will use bi-shadowing.

Theorem 3.4: Let both Φ and Ψ be bi-shadowing. Then the product system $\Phi\times\Psi$ is bi-shadowing.
Proof. Assume that for any $\beta > 0$ there exists $\alpha > 0$ and $0 \leq \delta \leq {}^{\beta}/_{\alpha}$, let $\{(x_{\mathbb{g}}, y_{\mathbb{g}}): \mathbb{g}\in\mathbb{G}\}$ be a δ-pseudo orbit of an action $\Phi\times\Psi$, and let $\mathbb{f}\times\mathbb{h}: \mathbb{G}\times\chi\times\Upsilon\to\chi\times\Upsilon$ be an action and satisfying.

$\sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right) \leq {}^{\beta}/_{\alpha} - \delta$

for $\mathbb{s}\in\mathbb{S}$ (6)

Since $\{(x_{\mathbb{g}}, y_{\mathbb{g}}): \mathbb{g}\in\mathbb{G}\}$ is a δ-pseudo orbit of an action $\Phi\times\Psi$ we have, for each $\mathbb{g}\in\mathbb{G}$, that

$max\{\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}}), \mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{g}\mathbb{s}})\} = \mathbb{D}\left(\left(\Phi(\mathbb{s}, x_{\mathbb{g}}), \Psi(\mathbb{s}, y_{\mathbb{g}})\right), (x_{\mathbb{g}\mathbb{s}}, y_{\mathbb{g}\mathbb{s}})\right) = \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}}, y_{\mathbb{g}})\right), (x_{\mathbb{g}\mathbb{s}}, y_{\mathbb{g}\mathbb{s}})\right) < \delta.$

Thus, $\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}}) < \delta$ and $\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{g}\mathbb{s}}) < \delta$, for $\mathbb{g}\in\mathbb{G}$, which implies that both $\{x_{\mathbb{g}}: \mathbb{g}\in\mathbb{G}\}$ of Φ and $\{y_{\mathbb{g}}: \mathbb{g}\in\mathbb{G}\}$ of Ψ are δ-pseudo trajectories. Since both Φ and Ψ are bi-shadowing, then for any actions $\mathbb{f}: \mathbb{G}\times\chi\to\chi$ and $\mathbb{h}: \mathbb{G}\times\Upsilon\to\Upsilon$ satisfying

$\mathbb{d}_{\chi_0}(\Phi,\mathbb{f}) = \sup_{x\in\chi}\{\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\} \leq \beta - \delta$ for $\mathbb{s}\in\mathbb{S}$, (7)

And

$\mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h}) = \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\} \leq \beta - \delta$ for $\mathbb{s}\in\mathbb{S}$, (8)

there exist true trajectories $\{w_{\mathbb{g}}: \mathbb{g}\in\mathbb{G}\}$ of $\mathbb{f}$ and $\{z_{\mathbb{g}}: \mathbb{g}\in\mathbb{G}\}$ of $\mathbb{h}$ such that

$\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) \leq \alpha\left(\delta + \mathbb{d}_{\chi_0}(\Phi,\mathbb{f})\right)$, for $\mathbb{g}\in\mathbb{G}$ (9)

And

$\mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}}) \leq \alpha\left(\delta + \mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h})\right)$, for $\mathbb{g}\in\mathbb{G}$ (10)

We may assume $\mathbb{d}_{\chi_0}(\Phi,\mathbb{f}) > \mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h})$, as the other direction can be treated similarly. Then By Lemma 3.3 with the relation (9) and (10) we get

$$\mathbb{D}\left(\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\left(w_{\mathbb{g}}, z_{\mathbb{g}}\right)\right)$$

$$\leq \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right)\right)\right) \leq \beta \text{ for } \mathbb{g}\in\mathbb{G}.$$

Finally, the sequence $\{(w_{\mathbb{g}}, z_{\mathbb{g}}): \mathbb{g}\in\mathbb{G}\}$ is a true orbit of $\mathbb{f}\times\mathbb{h}$ since

$$(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},\left(w_{\mathbb{g}}, z_{\mathbb{g}}\right)\right) = \left(\mathbb{f}(\mathbb{s}, w_{\mathbb{g}}), \mathbb{h}(\mathbb{s}, z_{\mathbb{g}})\right) = \left(w_{\mathbb{gs}}, z_{\mathbb{gs}}\right), \mathbb{g}\in\mathbb{G}$$

This means that $\Phi\times\Psi$ is bi-shadowing.
This ends the proof of Theorem 3.4.
For the converse direction of Theorem 3.2, we have the following partial result.

Theorem 3.5: Let Φ and Ψ be both actions and that the product system $\Phi\times\Psi$ is (α,β)-bi-shadowing. Then at least one of actions Φ and Ψ is (α,β)-bi-shadowing.

Proof. Let $\{x_{\mathbb{g}}: \mathbb{g}\in\mathbb{G}\}$ and $\{y_{\mathbb{g}}: \mathbb{g}\in\mathbb{G}\}$ be δ-pseudo trajectories of Φ and Ψ, respectively, with $0 \leq \delta \leq \beta$, and let the actions $\mathbb{f}: \mathbb{G}\times\chi\to\chi$ and $\mathbb{h}: \mathbb{G}\times\Upsilon\to\Upsilon$ satisfy the relations (2) and (3) respectively. Then for $\mathbb{g}\in\mathbb{G}$ we have the following estimates:

$$\mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right),\left(x_{\mathbb{gs}}, y_{\mathbb{gs}}\right)\right) = \mathbb{D}\left(\left(\Phi(\mathbb{s}, x_{\mathbb{g}}), \Psi(\mathbb{s}, y_{\mathbb{g}})\right),\left(x_{\mathbb{gs}}, y_{\mathbb{gs}}\right)\right)$$

$$= \max\{\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{gs}}), \mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{gs}})\} < \delta$$

Thus, the sequence $\{(x_{\mathbb{g}}, y_{\mathbb{g}}): \mathbb{g}\in\mathbb{G}\}$ is a δ-pseudo orbit of $\Phi\times\Psi$ and since $\Phi\times\Psi$ is (α,β)-bi-shadowing, for any action $\mathbb{f}\times\mathbb{h}: \mathbb{G}\times\chi\times\Upsilon\to\chi\times\Upsilon$ satisfying the relation (1) there exists a true orbit $\{(w_{\mathbb{g}}, z_{\mathbb{g}}): \mathbb{g}\in\mathbb{G}\}$ of $\mathbb{f}\times\mathbb{h}$ such that

$$\mathbb{D}\left(\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\left(w_{\mathbb{g}}, z_{\mathbb{g}}\right)\right)$$

$$\leq \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right)\right)\right), \mathbb{g}\in\mathbb{G}$$

So, for every $x\in\chi, y\in\Upsilon$, and using Lemma 3.1, we have

$$\max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\} \leq \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},\left(x_{\mathbb{g}}, y_{\mathbb{g}}\right)\right)\right)\right)$$
$$= \alpha\left(\delta + \sup_{x\in\chi, y\in\Upsilon} \max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\}\right)$$
$$= \alpha\left(\delta + \max\left\{\sup_{x\in\chi} \mathbb{d}_{\chi}\left(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)\right), \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right\}\right)$$

These estimates imply the following three cases.

Case 1: If $\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)) > \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then we have

$$\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) \leq \alpha\left(\delta + \sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\right), \mathbb{g}\in\mathbb{G}, \quad (11)$$

and hence Φ is (α,β)-bi-shadowing.

Case 2: If $\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)) < \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then we have

$$\mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}}) \leq \alpha\left(\delta + \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right), \mathbb{g}\in\mathbb{G}, \quad (12)$$

and hence Ψ is (α,β)-bi-shadowing.
Case 3: If $\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)) = \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then the relations in (11) and (12) both satisfied, and consequently, both Φ and Ψ are (α,β)-bi-shadowing.

Finally, it should be mentioned that both $\{w_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{f}$ and $\{z_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{h}$ are true trajectories, since

$$(\mathbb{f}(\mathbb{s}, w), \mathbb{h}(\mathbb{s}, z)) = (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (w_{\mathbb{g}}, z_{\mathbb{g}})\right) = (w_{\mathbb{g}\mathbb{s}}, z_{\mathbb{g}\mathbb{s}}), \mathbb{s} \in \mathbb{S}, \mathbb{g} \in \mathbb{G}.$$

This completes the proof of Theorem 3.5.
For the converse direction of Theorem 3.4, we have the following partial result.

Theorem 3.6: Let Φ and Ψ be both actions and that the product system $\Phi \times \Psi$ is bi-shadowing. Then at least one of actions Φ and Ψ is bi-shadowing.
Proof. Assume that for any $\beta > 0$ there exists $\alpha > 0$ and $0 \le \delta \le {}^{\beta}/_{\alpha}$, let $\{x_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ and $\{y_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ be δ-pseudo trajectories of Φ and Ψ, respectively, and let the actions $\mathbb{f}: \mathbb{G} \times \chi \to \chi$ and $\mathbb{h}: \mathbb{G} \times \Upsilon \to \Upsilon$ satisfy the relations (7) and (8) respectively. Then for $\mathbb{g} \in \mathbb{G}$ we have the following estimates:

$$\mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (x_{\mathbb{g}\mathbb{s}}, y_{\mathbb{g}\mathbb{s}})\right) = \mathbb{D}\left(\left(\Phi(\mathbb{s}, x_{\mathbb{g}}), \Psi(\mathbb{s}, y_{\mathbb{g}})\right), (x_{\mathbb{g}\mathbb{s}}, y_{\mathbb{g}\mathbb{s}})\right)$$

$$= max\{\mathbb{d}_{\chi}(\Phi(\mathbb{s}, x_{\mathbb{g}}), x_{\mathbb{g}\mathbb{s}}), \mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y_{\mathbb{g}}), y_{\mathbb{g}\mathbb{s}})\} < \delta.$$

Thus, the sequence $\{(x_{\mathbb{g}}, y_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ is a δ-pseudo orbit of $\Phi \times \Psi$ and since $\Phi \times \Psi$ is bi-shadowing, for any action $\mathbb{f} \times \mathbb{h}: \mathbb{G} \times \chi \times \Upsilon \to \chi \times \Upsilon$ satisfying the relation (6) there exists a true orbit $\{(w_{\mathbb{g}}, z_{\mathbb{g}}): \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{f} \times \mathbb{h}$ such that.

$$\mathbb{D}\left((x_{\mathbb{g}}, y_{\mathbb{g}})(w_{\mathbb{g}}, z_{\mathbb{g}})\right)$$

$$\le \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right), \mathbb{g} \in \mathbb{G}$$

So, for every $x \in \chi, y \in \Upsilon$, and using Lemma 3.1, we have.

$$\begin{aligned} max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\} &\le \alpha\left(\delta + \sup_{(x,y)\in\chi\times\Upsilon} \mathbb{D}\left((\Phi \times \Psi)\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right), (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (x_{\mathbb{g}}, y_{\mathbb{g}})\right)\right)\right) \\ &= \alpha\left(\delta + \sup_{x\in\chi, y\in\Upsilon} max\{\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}), \mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}})\}\right) \\ &= \alpha\left(\delta + max\left\{\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)), \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right\}\right) \end{aligned}$$

These estimates imply the following three cases.

Case 1: If $\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)) > \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then we have.

$$\mathbb{d}_{\chi}(x_{\mathbb{g}}, w_{\mathbb{g}}) \le \alpha\left(\delta + \sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x))\right) \le \beta, \mathbb{g} \in \mathbb{G}, \tag{13}$$

and hence Φ is bi-shadowing.
Case 2: If $\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)) < \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then we have

$$\mathbb{d}_{\Upsilon}(y_{\mathbb{g}}, z_{\mathbb{g}}) \le \alpha\left(\delta + \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}\right) \le \beta, \mathbb{g} \in \mathbb{G}, \tag{14}$$

and hence Ψ is bi-shadowing.
Case 3: If $\sup_{x\in\chi} \mathbb{d}_{\chi}(\Phi(\mathbb{s}, x), \mathbb{f}(\mathbb{s}, x)) = \sup_{y\in\Upsilon}\{\mathbb{d}_{\Upsilon}(\Psi(\mathbb{s}, y), \mathbb{h}(\mathbb{s}, y))\}$, then the relations in (13) and (14) both satisfied, and consequently, both Φ and Ψ are bi-shadowing.
Finally, it should be mentioned that both $\{w_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{f}$ and $\{z_{\mathbb{g}}: \mathbb{g} \in \mathbb{G}\}$ of $\mathbb{h}$ are true trajectories, since

$$(\mathbb{f}(\mathbb{s}, w), \mathbb{h}(\mathbb{s}, z)) = (\mathbb{f} \times \mathbb{h})\left(\mathbb{s}, (w_{\mathbb{g}}, z_{\mathbb{g}})\right) = (w_{\mathbb{g}\mathbb{s}}, z_{\mathbb{g}\mathbb{s}}), \mathbb{s} \in \mathbb{S}, \mathbb{g} \in \mathbb{G}.$$

This completes the proof of Theorem 3.6.

Theorem 3.7. Let Φ be (α,β)-bi-shadowing and Ψ is bi-shadowing. Then the product system $\Phi\times\Psi$ is (α,β)-bi-shadowing but not bi-shadowing.
Proof. Assume that the positive parameters α and β which choses by (α,β)-bi-shadowing and for any $\beta'>0$ there exists $\alpha'>0$ which define by Definition 2.3 there exist $0\le\delta\le{}^{\beta''}/_{\alpha''}$ when $\beta''=max\{\beta,\beta'\}$ and $\alpha''=min\{\alpha,\alpha'\}$, let $\{(x_{\mathbb{g}},y_{\mathbb{g}}):\mathbb{g}\in\mathbb{G}\}$ be a δ-pseudo orbit of an action $\Phi\times\Psi$, and let $\mathbb{f}\times\mathbb{h}:\mathbb{G}\times\chi\times\Upsilon\to\chi\times\Upsilon$ be an action satisfying

$$\sup_{(x,y)\in\chi\times\Upsilon}\mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}},y_{\mathbb{g}})\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(x_{\mathbb{g}},y_{\mathbb{g}})\right)\right)\le\beta''-\delta.$$

Since $\{(x_{\mathbb{g}},y_{\mathbb{g}}):\mathbb{g}\in\mathbb{G}\}$ is a δ-pseudo orbit of an action $\Phi\times\Psi$ we have, for each $\mathbb{g}\in\mathbb{G}$, that

$$max\{\mathbb{d}_\chi(\Phi(\mathbb{s},x_{\mathbb{g}}),x_{\mathbb{gs}}),\mathbb{d}_\Upsilon(\Psi(\mathbb{s},y_{\mathbb{g}}),y_{\mathbb{gs}})\}=\mathbb{D}\left(\left(\Phi(\mathbb{s},x_{\mathbb{g}}),\Psi(\mathbb{s},y_{\mathbb{g}})\right),(x_{\mathbb{gs}},y_{\mathbb{gs}})\right)=\mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}},y_{\mathbb{g}})\right),(x_{\mathbb{gs}},y_{\mathbb{gs}})\right)<\delta,\mathbb{s}\in\mathbb{S}.$$

Thus, $\mathbb{d}_\chi(\Phi(\mathbb{s},x_{\mathbb{g}}),x_{\mathbb{gs}})<\delta$ and $\mathbb{d}_\Upsilon(\Psi(\mathbb{s},y_{\mathbb{g}}),y_{\mathbb{gs}})<\delta$, for $\mathbb{s}\in\mathbb{S}$, $\mathbb{g}\in\mathbb{G}$, which implies that both $\{x_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ of Φ and $\{y_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ of Ψ are δ-pseudo trajectories.

Since Φ is (α,β)-bi-shadowing and Ψ is bi-shadowing, then for any actions $\mathbb{f}:\mathbb{G}\times\chi\to\chi$ and $\mathbb{h}:\mathbb{G}\times\Upsilon\to\Upsilon$ satisfying

$$\mathbb{d}_{\chi_0}(\Phi,\mathbb{f})=\sup_{x\in\chi}\{\mathbb{d}_\chi(\Phi(\mathbb{s},x),\mathbb{f}(\mathbb{s},x))\}\le\beta''-\delta \text{ for } \mathbb{s}\in\mathbb{S},$$

And

$$\mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h})=\sup_{y\in\Upsilon}\{\mathbb{d}_\Upsilon(\Psi(\mathbb{s},y),\mathbb{h}(\mathbb{s},y))\}\le\beta''-\delta \text{ for } \mathbb{s}\in\mathbb{S},$$

there exist true trajectories $\{w_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ of $\mathbb{f}$ and $\{z_{\mathbb{g}}:\mathbb{g}\in\mathbb{G}\}$ of $\mathbb{h}$ such that

$$\mathbb{d}_\chi(x_{\mathbb{g}},w_{\mathbb{g}})\le\alpha''\left(\delta+\mathbb{d}_{\chi_0}(\Phi,\mathbb{f})\right), \text{ for } \mathbb{g}\in\mathbb{G} \qquad (15)$$

And

$$\mathbb{d}_\Upsilon(y_{\mathbb{g}},z_{\mathbb{g}})\le\alpha''\left(\delta+\mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h})\right), \text{ for } \mathbb{g}\in\mathbb{G} \ (16)$$

We may assume $\mathbb{d}_{\chi_0}(\Phi,\mathbb{f})>\mathbb{d}_{\Upsilon_0}(\Psi,\mathbb{h})$, as the other direction can be treated similarly. Then By Lemma 3.3 with the relation (15) and (16) we get

$$\mathbb{D}\left((x_{\mathbb{g}},y_{\mathbb{g}})(w_{\mathbb{g}},z_{\mathbb{g}})\right)$$

$$\le\alpha''\left(\delta+\sup_{(x,y)\in\chi\times\Upsilon}\mathbb{D}\left((\Phi\times\Psi)\left(\mathbb{s},(x_{\mathbb{g}},y_{\mathbb{g}})\right),(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(x_{\mathbb{g}},y_{\mathbb{g}})\right)\right)\right)\le\beta'' \text{ for } \mathbb{g}\in\mathbb{G}.$$

Finally, the sequence $\{(w_{\mathbb{g}},z_{\mathbb{g}}):\mathbb{g}\in\mathbb{G}\}$ is a true orbit of $\mathbb{f}\times\mathbb{h}$ since

$$(\mathbb{f}\times\mathbb{h})\left(\mathbb{s},(w_{\mathbb{g}},z_{\mathbb{g}})\right)=\left(\mathbb{f}(\mathbb{s},w_{\mathbb{g}}),\mathbb{h}(\mathbb{s},z_{\mathbb{g}})\right)=(w_{\mathbb{gs}},z_{\mathbb{gs}}),\mathbb{g}\in\mathbb{G}$$

From our assumptions that [β is positive parameters] and [for any $\beta'>0$] we conclude that $\beta>\beta'$ and then $\beta''=max\{\beta,\beta'\}=\beta$, this means that $\Phi\times\Psi$ is (α,β) bi-shadowing but not bi-shadowing.
This ends the proof of Theorem 3.7.

Corollary 3.8. Let Φ be (α,β)-bi-shadowing and the product system $\Phi\times\Psi$ is bi-shadowing. Then Ψ must be bi-shadowing.
Proof. Since Φ is (α,β)-bi-shadowing, then it is not bi-shadowing, then By Theorem 3.6 Ψ must be bi-shadowing.

Multiple Product System
Theorem 4.1. For $n>2$, Let the actions $\Phi^1,\Phi^2,\ldots,\Phi^n$ be (α,β)-bi-shadowing. Then the multiple product system $\Phi^1\times\Phi^2\times\ldots\times\Phi^n$ is (α,β)-bi-shadowing.
Proof. Assume that $n=3$ then we have the actions Φ^1,Φ^2,Φ^3 are (α,β)-bi-shadowing, by using Theorem 3.2 we get the action $(\Phi^1\times\Phi^2)$ is (α,β)-bi-shadowing, again using Theorem 3.2 of the actions $(\Phi^1\times\Phi^2)$ and Φ^3 we get $\Phi^1\times\Phi^2\times\Phi^3$ is (α,β)-bi-shadowing.
By induction, the theorem can easily be proven when $n>3$.

This ends the proof of Theorem 4.1.
Using the same proof method above but based on Theorem 3.4, the following theorem can be proven.

Theorem 4.2: For $n > 2$, Let all actions $\Phi^1, \Phi^2, \dots, \Phi^n$ be bi-shadowing. Then the multiple product system $\Phi^1 \times \Phi^2 \times \dots \times \Phi^n$ is bi-shadowing.

Theorem 4.3: For $n > 2$, Let that $\Phi^1, \Phi^2, \dots, \Phi^n$ be actions and that the multiple product system $\Phi^1 \times \Phi^2 \times \dots \times \Phi^n$ is (α, β)-bi-shadowing. Then at least one of actions $\Phi^1, \Phi^2, \dots, \Phi^n$ is (α, β)-bi-shadowing.
Proof. Assume that $n = 3$ then we have the actions Φ^1, Φ^2, Φ^3 are (α, β)-bi-shadowing, also by using Theorem 3.2 the action $(\Phi^1 \times \Phi^2)$ is (α, β)-bi-shadowing.
Now by using Theorem 3.5 we get at least one of actions $(\Phi^1 \times \Phi^2)$ or Φ^3 is (α, β)-bi-shadowing, then either Φ^3 or $(\Phi^1 \times \Phi^2)$ again using Theorem 3.5 of the actions $(\Phi^1 \times \Phi^2)$ then at least one of actions Φ^1 or Φ^2 is (α, β)-bi-shadowing. Therefore at least one of actions Φ^1, Φ^2, Φ^3 is (α, β)-bi-shadowing
By induction, the theorem can easily be proven when $n > 3$.
This ends the proof of Theorem 4.3.
Using the same proof method above but based on Theorem 3.4 and Theorem 3.6, the following theorem can be proven.

Theorem 4.4: For $n > 2$, Let $\Phi^1, \Phi^2, \dots, \Phi^n$ be actions and that the multiple product system $\Phi^1 \times \Phi^2 \times \dots \times \Phi^n$ is bi-shadowing. Then at least one of actions $\Phi^1, \Phi^2, \dots, \Phi^n$ is bi-shadowing.

Theorem 4.5: For $n > 2$, Let all $\Phi^1, \Phi^2, \dots, \Phi^n$ be actions if one or more of them is (α, β)-bi-shadowing and all other are bi-shadowing. Then the multiple product system $\Phi^1 \times \Phi^2 \times \dots \times \Phi^n$ is (α, β)-bi-shadowing but not bi-shadowing.
Proof. Assume that $n = 3$

Case 1: Assume that Φ^1, Φ^2 is (α, β)-bi-shadowing and Φ^3 are bi-shadowing, by using Theorem 3.2 the action $(\Phi^1 \times \Phi^2)$ is (α, β)-bi-shadowing. By using Theorem 3.7 we get product system $(\Phi^1 \times \Phi^2) \times \Phi^3 = \Phi^1 \times \Phi^2 \times \Phi^3$ is (α, β)-bi-shadowing but not bi-shadowing.

Case 2: Assume that Φ^1 is (α, β)-bi-shadowing and Φ^2, Φ^3 are bi-shadowing, by using Theorem 3.4 the action $(\Phi^2 \times \Phi^3)$ is bi-shadowing. By using Theorem 3.7 we get product system $\Phi^1 \times (\Phi^2 \times \Phi^3) = \Phi^1 \times \Phi^2 \times \Phi^3$ is (α, β)-bi-shadowing but not bi-shadowing.
For $n > 3$ we can use Theorem 4.1 and Theorem 4.2 with Case 1 or Case 2.
This ends the proof of Theorem 4.5.

Corollary 4.6. For $n > 2$, Let all $\Phi^1, \Phi^2, \dots, \Phi^n$ be actions if all of them except one are (α, β)-bi-shadowing and the multiple product system $\Phi^1 \times \Phi^2 \times \dots \times \Phi^n$ is bi-shadowing. Then this actions must be bi-shadowing.
Proof. Assume that $n = 3$

Case 1: Assume that Φ^1, Φ^2 is (α, β)-bi-shadowing and Φ^3 is an action, by using Theorem 3.2 the action $(\Phi^1 \times \Phi^2)$ is (α, β)-bi-shadowing. By using Corollary 3.8 we get Φ^3 must be bi-shadowing.

Case 2: For $n > 3$ we can use Theorem 4.1 with Case 1.
This ends the proof of Corollary 4.6.

Conclusion

From the presented study, it is evident that the exploration of bi-shadowing properties within dynamical systems, particularly within group spaces, has yielded valuable insights. By defining and analyzing the (α,β)-bi-shadowing and bi-shadowing concepts, this paper contributes to the theoretical framework of dynamical systems, expanding the scope of shadowing theory.
The theorems established in this paper provide essential connections between different types of bi-shadowing properties within group spaces. The results highlight the interplay between these properties and demonstrate their implications for the behavior of dynamic systems.
Furthermore, the investigation into product systems adds depth to our understanding, illustrating how the bi-shadowing property can be preserved or altered when considering compositions of dynamical systems.
This paper not only enriches the theoretical foundation of shadowing theory but also offers practical insights into the behavior of dynamical systems, particularly in group spaces. The findings presented here open avenues for further research in this area, with potential applications across various domains where dynamical systems play a crucial role.

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