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**Hind H Hussien**
Department of Mathematics, College of Education for Women, University of Tikrit, Tikrit, Iraq

**Hiba O Mousa**
Department of Mathematics, College of Education for Women, University of Tikrit, Tikrit, Iraq

**Corresponding Author:**
**Hind H Hussien**
Department of Mathematics, College of Education for Women, University of Tikrit, Tikrit, Iraq

# On $D_{\sigma^{\#}}$: Open Sets in $q$-Topological space

**Hind H Hussien and Hiba O Mousa**

**Abstract**
This research study introduces a novel idea of open sets, known as $D_{\sigma^{\#}}$-open, which is mostly based on quad topology. Furthermore, we not only extend this inclusive collection, but we also introduce the encompassing, internal, and external aspects of this collection, an accompanied by specific instances and significant principles.

**Keywords:** Sets in $q$-Topological space, quad topology, encompassing, internal, and external aspects

## Introduction

Many studies have presented a new types of open sets, some of which are based many on topological spaces, including but not limit to: $(\wedge, b)$-open sets introduced in 2022 by [1], also *-open sets in 2019 by [2], an open upper-semi-cont. topologies which introduced in 2018 and shown usefully and its properties by [3], in additionally to open sets that were used to build graphs models, clarify the topological structure for distributing this graphs to subsets by [4], and in the same your introduced delta open sets ($\delta$-open) by [5]. These and the other open sets helping the many researchers in to understood and introduced a new topological spaces and its properties.

In addition the open sets, must address the topological space that we will rely on in building the structure of the new open set. Quad topology was presented for the first time by Sharma *et al.*, where the researchers formed a new open set called $q$-open and relied on it as [6]. Many studies introduced on this topology as: fuzzy $q$-open and fuzzy $q$-$b$-open sets also $q$-cont. and $q$-$b$-cont. functions by [7], as well as presenting a study of a new function for quad topology space, it designs new and different types of sets [8]. Moreover, Delving deeper into the four-part neutrosophic topology on four-part neutrosophic groups, identifying different types of sets such as semi-open neutrosophic sets, semi-open sets, open $B$, and open alpha sets within the four-part neutrosophic topological spaces [9]. Pang *et al*. have proposed a topological structure for a quadrupole transmitter consisting of a first topology and a second topology, where the first topology includes a first bridge, a second phase shift element, and a third phase shift elements [10]. Collectively, these studies contribute to the exploration and understanding of different types of structures. Topology and its properties.

The study aims to find and present a new open set called $D_{\sigma^{\#}}$-open, in addition to providing generalizations for this set with some basic concepts, examples, and theorems for this. The study consisted of four sections. The first section contained the introduction to the study, while the second section contained some basic concepts. The third section contained the proposed new group, its generalizations, and illustrative examples. The fourth and final section included the most important conclusions reached by the study.

## Fundamental concepts

**Definition (2.1) [11]:** Consider a nonempty set $X$ with four universal topologies, denoted as $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$, on $X$. The subset $A$ of space $X$ is said to be quad-open ($q$-open) if it meets the requirements $A \subset \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$. The term used to describe the complement of set A is "$q$-closed". The set $X$ is associated with four topologies known as $q$-topological spaces $(X, \mathcal{T}_q)$, where $q = 1,2,3,4$. Open sets in a topological space fulfil all the axioms of topology.

**Definition (2.2) [11]:** The $q$-interior of a subset $A$, represented as $q_{int}(A)$, is the combination of all $q$-open sets that are included in $A$. The $q$-closure of set $A$, represented as $q_{cl}(A)$. The set $q_{cl}(A)$ is defined as an intersection of all $q$-closed sets including $A$.

**Remark (2.3) [11]:** The union of $q$-open sets stays $q$-open. Moreover, the intersection of every random collection of $q$-closed sets is also $q$-closed.

**Definition (2.4) [12]:** A set $A$ in a topological space is defined as generalised closed (abbreviated as $g$-closed) if and only if the closure of $A$ ($cl(A)$) is a subset of any set $U$ whenever $A$ is a subset of $U$ and $U$ is open.

**Definition (2.5) [12]:** $A$ is the set defined as a $g^*$-closed set ($clg^*(A)$) if and only if there is a closed set $F$ that includes $A$, in a manner that whenever $A$ is a subset of $U$ and $U$ is an open set, then $F$ is also a subset of $U$.

**Definition (2.6) [12]:** Every closed set is guaranteed to be $g^*$-closed, however, the converse is not always true.

**$D_{\sigma^\#}$-open sets**

**Definition (3.1):** A subset of $(\mathrm{X}, \tau_{\mathrm{q}})$ is purported to be $D_{\sigma^\#}$-open if $A \subset T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A)\big)\right)\right)$; where $q = 1,2,3,4$.

The collection of all $D_{\sigma^\#}$-open sets in the topology space $(X, \tau_q)$ is represented by $D_{\sigma^\#}$-$O(X)$.

**Definition (3.2):** The complement of $D_{\sigma^\#}$-open in $q$-topology space $(X, \tau_q)$ is purported to be $D_{\sigma^\#}$-closed sets, denoted by $D_{\sigma^\#}$-$C(X)$.

**Example (3.3):** Let $X = \{a, m, c\}$, where $\tau_1 = \{X, \emptyset, \{\mathrm{c}\}, \{a, c\}\}$ , $\tau_2 = \{X, \emptyset, \{\mathrm{a}\}\}$, $\tau_2 clg^* = \{X, \emptyset, \{\mathrm{m}\}, \{c\}, \{a, \mathrm{m}\}, \{a, \mathrm{c}\}, \{m, c\}\}$, $\tau_3 = \{X, \emptyset, \{\mathrm{c}\}, \{m, c\}\}$, $\tau_4 = \{X, \emptyset, \{\mathrm{b}, c\}\}$, $\tau_4 clg^* = \{X, \emptyset, \{\mathrm{a}\}, \{a, m\}, \{a, c\}\}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}$-$O(X) = \{X, \emptyset, \{c\}, \{m, c\}\}$. Hence the set of all $D_{\sigma^\#}$-closed set in $X$ is $D_{\sigma^\#}$-$C(X) = \{X, \emptyset, \{a\}, \{a, m\}\}$

**Theorem (3.4):** The collection of every one of $D_{\sigma^\#}$-open sets forms a topology space.

**Proof**

1. $\emptyset$, and $X$ clearly $D_{\sigma^\#}$-open set.
2. Let $S_1$, $S_2$ are $D_{\sigma^\#}$-open sets, $S_1 = T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A)\big)\right)\right)$, and $S_2 = T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(B)\big)\right)\right)$, such as $A, B \subset X$,
   Then $S_1 \cap S_2 = T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A)\big)\right)\right) \cap T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(B)\big)\right)\right) = T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A \cap B)\big)\right)\right) \Longrightarrow S_1 \cap S_2$ is $D_{\sigma^\#}$-open set.
3. Let $S_\lambda$, $\lambda \in \wedge$, where $S_\lambda = T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A)\big)\right)\right), A_\lambda \subset X$, then $\cup_\lambda S_\lambda = \cup_\lambda T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A_\lambda)\big)\right)\right) = T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(\cup_\lambda A_\lambda)\big)\right)\right)$, given that $\cup_\lambda A_\lambda \in X \Longrightarrow \cup_\lambda T_1 - int\left(T_2 - clg^*\left(T_3 - int\big(T_4 - clg^*(A_\lambda)\big)\right)\right)$ is $D_{\sigma^\#}$-open set $\Longrightarrow \cup_\lambda S_\lambda$ is $D_{\sigma^\#}$-open set.

Hence the family of all $D_{\sigma^\#}$-open sets is topological space

**Example (3.5):** Let $X = \{a, m, c\}$, where $\tau_1 = \{X, \emptyset, \{a\}\}$ , $\tau_2 = \{X, \emptyset, \{a\}, \{a, m\}\}$, $\tau_2 clg^* = \{X, \emptyset, \{c\}, \{a, c\}, \{m, c\}\}$, $\tau_3 = \{X, \emptyset, \{a\}, \{a, c\}\}$, $\tau_4 = \{X, \emptyset, \{a, m\}\}$, $\tau_4 clg^* = \{X, \emptyset, \{c\}, \{m, c\}, \{a, c\}\}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}$-$O(X) = \{X, \emptyset, \{a\}, \{a, m\}\}$

**Remark (3.6):** The set of $D_{\sigma^\#}$-open sets is a q-open set, but the reverse doesn't hold valid.

**Example (3.7):** Let $X = \{a, m, c\}$, where $\tau_1 = \{X, \emptyset, \{a\}, \{m\}, \{a, m\}\}$, $\tau_2 = \{X, \emptyset, \{c\}\}$, $\tau_2 clg^* = \{X, \emptyset, \{a\}, \{m\}, \{a, m\}, \{a, c\}, \{m, c\}\}$, $\tau_3 = \{X, \emptyset, \{m\}, \{a, m\}\}$, $\tau_4 = \{X, \emptyset, \{a, m\}\}$, $\tau_4 clg^* = \{X, \emptyset, \{c\}, \{m, c\}, \{a, c\}\}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}$-$O(X) = \{X, \emptyset, \{m\}, \{a, m\}\}$, $q$-$O(X) = \{X, \emptyset, \{a\}, \{m\}, \{c\}, \{a, m\}\}$, $\{c\}$ is $q$-$O(X)$, but $\{c\}$ is not $D_{\sigma^\#}$. It's clearly $D_{\sigma^\#}$-open sets is a $q$-open.

**Definition (3.8):** A $q$-topological space $(X, \tau_q)$ is referred to as discrete $q$-topological with respect to $D_{\sigma^\#}$-open if $D_{\sigma^\#}$-$O(X)$ contains all subsets on $X$.

**Example (3.9):** Let $X = \{a, m, c\}$, where $\tau_1 = \{X, \emptyset, \{a\}, \{m\}, \{c\}, \{a, m\}, \{a, c\}, \{m, c\}\}$, $\tau_2 = \{X, \emptyset, \{m\}\}$, $\tau_2 clg^* = \{X, \emptyset, \{a\}, \{c\}, \{a, m\}, \{a, c\}, \{m, c\}\}$, $\tau_3 = \{X, \emptyset, \{a\}, \{m\}, \{a, m\}, \{m, c\}\}$, $\tau_4 = \{X, \emptyset, \{c\}, \{a, c\}\}$, $\tau_4 clg^* = \{X, \emptyset, \{m\}, \{a, m\}, \{m, c\}\}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}$-$O(X) = \{X, \emptyset, \{a\}, \{m\}, \{c\}, \{a, m\}, \{a, c\}, \{m, c\}\}$. It's clearly that $D_{\sigma^\#}$-open set is discrete $q$-topological space.

**Definition (3.10):** A $q$-topological space $(X, \tau_q)$ is called indiscrete $q$-topological with respect to $D_{\sigma^\#}$-open if $D_{\sigma^\#}$-$O(X) = \{X, \emptyset\}$

**Example (3.11):** Let $X = \{a, m, c, d\}$, where $\tau_1 = \{X, \emptyset, \{a\}\}$ , $\tau_2 = \{X, \emptyset, \{m\}\}$, $\tau_2 clg^* = \begin{Bmatrix} X, \emptyset, \{a\}, \{c\}, \{d\}, \{a, m\}, \{a, c\}, \{a, d\}, \{m, c\}, \{m, d\} \\ \{c, d\}, \{a, m, c\}, \{a, m, d\}, \{m, c, d\}, \{a, c, d\} \end{Bmatrix}$, $\tau_3 = \{X, \emptyset, \{c\}\}$, $\tau_4 = \{X, \emptyset, \{d\}\}$, $\tau_4 clg^* = \begin{Bmatrix} X, \emptyset, \{a\}, \{c\}, \{m\}, \{a, m\}, \{a, c\}, \{a, d\}, \{m, c\}, \{m, d\} \\ \{c, d\}, \{a, m, c\}, \{m, c, d\}, \{a, c, d\}, \{a, m, d\} \end{Bmatrix}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}$-$O(X) = \{X, \emptyset\}$. It's clearly that $D_{\sigma^\#}$-open set is indiscrete $q$-topological space.

**Definition (3.12):** A subset $A$ for a $q$-topological space $(X, \tau_q)$ is a $D_{\sigma^\#}$-neighborhood of the point $x \in X$ if there occurs a $D_{\sigma^\#}$-open set S that corresponds to $x \in S \subseteq A$. The set of all $D_{\sigma^\#}$-neighborhoods of the point x is denoted as $D_{\sigma^\#}$-$N(x)$.

**Example (3.13):** Referring to the given example (3.7), we get: $D_{\sigma^\#}$-$O(X) = \{X, \emptyset, \{m\}, \{a, m\}\}$. If we take $a \in X$, then $D_{\sigma^\#}$-$N(a) = \{X, \emptyset, \{a, m\}\}$

**Definition (3.14):** In a q-topological space $(X, \tau_q)$, the $D_{\sigma^\#}$-interior of a subset $A$ is defined as the union of all $D_{\sigma^\#}$-open subsets that are contained in $A$. This $D_{\sigma^\#}$-interior is denoted as $D_{\sigma^\#}$-$int(A)$. Therefore, $D_{\sigma^\#}$-$int(A)$ refers to the greatest $D_{\sigma^\#}$-open subset of $A$.

**Example (3.15):** From the example (3.3), we get: $D_{\sigma^\#}$-$O(X) = \{X, \emptyset, \{c\}, \{m, c\}\}$. If we take $A = \{a, c\}$, then $D_{\sigma^\#}$-$int(A) = \{\{c\}\}$

**Theorem (3.16):** Let $(X, \tau_q)$ be a $q$-topological space and $A \subset X$, then $D_{\sigma^\#}$-$int(A) = \cup \{S \in D_{\sigma^\#} - O(X)\}; S \subset A$.
Proof : Let $x \in D_{\sigma^\#}$-$int(A)$ iff $A$ is a $D_{\sigma^\#}$-nhd iff $\exists D_{\sigma^\#}$-open set $S$ s.t $x \in S \subset A$ iff $x \in \cup \{S \in D_{\sigma^\#} - O(X)\}$, $S \subset A$. Hence $int(A) = \cup \{S \in D_{\sigma^\#} - O(X)\}, S \subset A$.

**Theorems (3.17):** Consider a q-topological space $(X, \tau_q)$ and let $A$ be a subset of $X$. Then,
1. $D_{\sigma^\#}$-$int(A)$ is an $D_{\sigma^\#}$-open set.
2. $D_{\sigma^\#}$-$int(A)$ is the greatest $D_{\sigma^\#}$-open set enclosed within $A$.
3. $A$ is $D_{\sigma^\#}$-open iff $D_{\sigma^\#}$-$int(A) = A$

**Proof**
1. Let $x$ be any arbitrary point of $D_{\sigma^\#}$-$int(A)$. Then $x$ is $D_{\sigma^\#}$-interior point of $A$ hence by definition. $A$ is a $D_{\sigma^\#}$-nhd of $X$, then There is a presence of a $D_{\sigma^\#}$-open set $S$ in a manner that $x \in S \subset A$. Given that $S$ is $D_{\sigma^\#}$-open set, it's a $D_{\sigma^\#}$-nhd each of its points so $A$ is also $D_{\sigma^\#}$-nhd of each point of $S$. It follows that every point of $S$ is a $D_{\sigma^\#}$-interior point of $A$ so that $S \subset D_{\sigma^\#}$-$int(A)$. Thus its shown that to each $x \in D_{\sigma^\#}$-$int(A)$, There is a presence of a $D_{\sigma^\#}$-open set $S$ in a manner that $x \in G \subset D_{\sigma^\#}$-$int(A)$.
   Hence $D_{\sigma^\#}$-$int(A)$ is a $D_{\sigma^\#}$-nhd of each its point and consequently $D_{\sigma^\#}$-$int(A)$ is $D_{\sigma^\#}$-open.
2. Let $S$ be any $D_{\sigma^\#}$-open subset of $A$ and let $x \in S$. So that $x \in S \subset A$, given that $S$ is $D_{\sigma^\#}$-open, $A$ is $D_{\sigma^\#}$-nhd of $X$ and consequently $x$ is a $D_{\sigma^\#}$-interior point of $A$.
   Hence $x \in D_{\sigma^\#}$-$int(A)$. Hence $D_{\sigma^\#}$-$int(A)$ contains every $D_{\sigma^\#}$-open subset of $A$ thus we have shown that $x \in S$, then $x \in D_{\sigma^\#}$-$int(A)$ and $S \subset D_{\sigma^\#}$-$int(A) \subset A$ therefore the largest $D_{\sigma^\#}$-$int(A)$ is $D_{\sigma^\#}$-open subset of $A$.
3. Let $A = D_{\sigma^\#}$-$int(A)$, by (1) $D_{\sigma^\#}$-$int(A)$ is $D_{\sigma^\#}$-open set and therefore $A$ is also $D_{\sigma^\#}$-open
   Conversely, let $A$ be a $D_{\sigma^\#}$-open set. Then $A$ is surly identical with the largest $D_{\sigma^\#}$-open subset of $A$, but by(2), $D_{\sigma^\#}$-$int(A)$ is the greatest $D_{\sigma^\#}$-open subset of $A$ hence $D_{\sigma^\#}$-$int(A) = A$

**Theorems (3.18):** Consider a q-topological space $(X, \tau_q)$ and let $P$ and $B$ be subsets of $X$. Then:
1. $D_{\sigma^\#}$-$int(X) = X$, $D_{\sigma^\#}$-$int(\emptyset) = \emptyset$.
2. $D_{\sigma^\#}$-$int(P) \subset P$.
3. If $P \subset B$, subsequently $D_{\sigma^\#}$-$int(P) \subset D_{\sigma^\#}$-$int(B)$.
4. $D_{\sigma^\#}$-$int(P \cap B) \subset D_{\sigma^\#}$-$int(P) \cap D_{\sigma^\#}$-$int(B)$
5. $D_{\sigma^\#}$-$int(P) \cup D_{\sigma^\#}$-$int(B) \subset D_{\sigma^\#}$-$int(P \cup B)$
6. $D_{\sigma^\#}$-$int(D_{\sigma^\#}$-$int(P)) \subset D_{\sigma^\#}$-$int(P)$

**Proof**
1. Given that $X$ and $\emptyset$ are $D_{\sigma^\#}$-open sets, so by [theorem (3.17), part 3] $D_{\sigma^\#}$-$int(X) = X$, $D_{\sigma^\#}$-$int(\emptyset) = \emptyset$.

2. If $x \in D_{\sigma^\#}\text{-}int(P)$, subsequently $x$ is a $D_{\sigma^\#}$-interior point of $P$, and $P$ is $D_{\sigma^\#}$-nhd of $X$, $x \in P$, hence $D_{\sigma^\#}\text{-}int(P) \subset P$
3. Let $x \in D_{\sigma^\#}\text{-}int(P)$, then $x$ is $D_{\sigma^\#}$-interior point of $P$, and $P$ is $D_{\sigma^\#}$-nhd of $X$, so $P \subset B$, then $B$ also is $D_{\sigma^\#}$-nhd of $X$, this implies that $x \in D_{\sigma^\#}\text{-}int(B)$, hence $D_{\sigma^\#}\text{-}int(P) \subset D_{\sigma^\#}\text{-}int(B)$.
4. Given that $P \cap B \subset P$, and $P \cap B \subset B$, by (3) above we have $D_{\sigma^\#}\text{-}int(P \cap B) \subset D_{\sigma^\#}\text{-}int(P)$, $D_{\sigma^\#}\text{-}int(P \cap B) \subset D_{\sigma^\#}\text{-}int(B)$, hence $D_{\sigma^\#}\text{-}int(P \cap B) \subset D_{\sigma^\#}\text{-}int(P) \cap D_{\sigma^\#}\text{-}int(B)$
5. So by (3) above we have $P \subset P \cup B$ then $D_{\sigma^\#}\text{-}int(P) \subset D_{\sigma^\#}\text{-}int(P \cup B)$, $B \subset P \cup B$ then $D_{\sigma^\#}\text{-}int(B) \subset D_{\sigma^\#}\text{-}int(P \cup B)$. Hence $D_{\sigma^\#}\text{-}int(P) \cup D_{\sigma^\#}\text{-}int(B) \subset D_{\sigma^\#}\text{-}int(P \cup B)$
6. By [theorem (3.17) part(1)] then $D_{\sigma^\#}\text{-}int(P)$ is a $D_{\sigma^\#}$-open set and [ by the part (3) from this theorem ] hence $D_{\sigma^\#}\text{-}int\left(D_{\sigma^\#} - int(P)\right) = D_{\sigma^\#}\text{-}int(P)$

**Remark (3.19)**
1. The converse of $[D_{\sigma^\#}\text{-}int(P) \subset P]$ is not true in general
2. The converse of part [If $P \subset B$, then $D_{\sigma^\#}\text{-}int(P) \subset D_{\sigma^\#}\text{-}int(B)$] is not true in general.
3. The converse of $[D_{\sigma^\#}\text{-}int(P) \cup D_{\sigma^\#}\text{-}int(B) \subset D_{\sigma^\#}\text{-}int(P \cup B)]$ is not true in general.

**Definition (3.20):** A subset $P$ of a $q$-toplogical space $(X, \tau_q)$, a point $x \in X$ is $D_{\sigma^\#}$-exterior point of $P$ iff is an $D_{\sigma^\#}$-interior point of the complement of $P$, in a manner that $x \in S \subset P^c$, or $x \in S$ and $S \cap P = \emptyset$, where $S$ is $D_{\sigma^\#}$-open set. The set of all $D_{\sigma^\#}$-exterior point of $P$ is denoted by $D_{\sigma^\#}\text{-}ext(P)$, i.e $D_{\sigma^\#}\text{-}ext(P) = D_{\sigma^\#}\text{-}int(P^c)$

**Example (3.21):** Let $X = \{a, m, c\}$, where $\tau_1 = \{X, \emptyset, \{m\}, \{c\}, \{m, c\}\}$ , $\tau_2 = \{X, \emptyset, \{a\}, \{a, m\}\}$, $\tau_2 clg^* = \{X, \emptyset, \{c\}, \{a, c\}, \{m, c\}\}$, $\tau_3 = \{X, \emptyset, \{c\}, \{a, c\}\}$, $\tau_4 = \{X, \emptyset, \{a, m\}\}$, $\tau_4 clg^* = \{X, \emptyset, \{c\}, \{a, c\}, \{m, c\}\}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}\text{-}O(X) = \{X, \emptyset, \{c\}, \{a, m\}\}$. If we take $P = \{a\}$, then $D_{\sigma^\#}\text{-}ext(P) = D_{\sigma^\#}\text{-}int(P^c) = D_{\sigma^\#}\text{-}int(\{m, c\}) = \{c\}$

**Theorem (3.22):** Consider a $q$-topological space $(X, \tau_q)$ and let P be a subset of $X$, subsequently $D_{\sigma^\#}\text{-}ext(P) = \cup \{S \in D_{\sigma^\#} - O(X)\}; S \in P^c$.

Proof : By definition (2.20) $D_{\sigma^\#}\text{-}ext(P) = D_{\sigma^\#}\text{-}int(P^c)$ but by theorem (3.16) $D_{\sigma^\#}\text{-}int(P^c) = \cup \{S \in D_{\sigma^\#} - O(X)\}; S \in P^c$. Hence$D_{\sigma^\#}\text{-}ext(P) = \cup \{S \in D_{\sigma^\#} - O(X)\}; S \in P^c$

**Remark (3.23):** Given that $D_{\sigma^\#}\text{-}ext(P) = D_{\sigma^\#}\text{-}int(P^c)$, [by theorem (3.16)] we get $D_{\sigma^\#}\text{-}ext(P)$ is $D_{\sigma^\#}$-open set and it largest $D_{\sigma^\#}$-open set contained $P^c$.

**Theorems (2.24):** Consider a q-topological space $(X, \tau_q)$ and let $P$ and $B$ be subsets of $X$, subsequently:
1. $D_{\sigma^\#}\text{-}ext(X) = \emptyset$, $D_{\sigma^\#}\text{-}ext(X) = \emptyset$.
2. $D_{\sigma^\#}\text{-}ext(P) \subset P^c$.
3. $D_{\sigma^\#}\text{-}ext(D_{\sigma^\#}\text{-}ext(P^c)) = D_{\sigma^\#}\text{-}ext(P)$
4. If $P \subset B$, subsequently $D_{\sigma^\#}\text{-}ext(B) \subset D_{\sigma^\#}\text{-}ext(P)$.
5. $D_{\sigma^\#}\text{-}ext(P) \subset D_{\sigma^\#}\text{-}ext\left(D_{\sigma^\#} - ext(P)\right)$
6. $D_{\sigma^\#}\text{-}ext(P \cup B) \subset D_{\sigma^\#}\text{-}ext(P) \cap D_{\sigma^\#}\text{-}ext(B)$

**Proof**
1. $D_{\sigma^\#}\text{-}ext(X) = D_{\sigma^\#}\text{-}int(X^c) = D_{\sigma^\#}\text{-}int(\emptyset) = \emptyset$
   $D_{\sigma^\#}\text{-}ext(\emptyset) = D_{\sigma^\#}\text{-}int(\emptyset^c) = D_{\sigma^\#}\text{-}int(X) = X$
2. $D_{\sigma^\#}\text{-}ext(P) = D_{\sigma^\#}\text{-}int(P^c) \subset P^c$, [by (2) of theorem (3.18)]
3. $D_{\sigma^\#}\text{-}ext(\left(D_{\sigma^\#} - ext(P)\right)^c) = D_{\sigma^\#}\text{-}ext\left(D_{\sigma^\#} - int((P^c)^c)^c\right) = D_{\sigma^\#}\text{-}int\left(\left(\left(D_{\sigma^\#} - int(P^c)\right)^c\right)\right)^c = D_{\sigma^\#}\text{-}int\left(D_{\sigma^\#} - int(P^c)\right)^c$ given that $P^{cc} = P$, and $D_{\sigma^\#}\text{-}int\left(D_{\sigma^\#} - int(P)\right) = D_{\sigma^\#}\text{-}int(P) = D_{\sigma^\#}\text{-}int(P^c) = D_{\sigma^\#}\text{-}ext(P)$
4. $P \subset B$, then $B^c \subset P^c$, then $D_{\sigma^\#}\text{-}int(B^c) \subset D_{\sigma^\#}\text{-}int(P^c)$, then $D_{\sigma^\#}\text{-}int(B) \subset D_{\sigma^\#}\text{-}int(P)$.
5. By(2), we have $D_{\sigma^\#}\text{-}ext(P) \subset P^c$. Then (4) gives $D_{\sigma^\#}\text{-}ext(P^c) \subset D_{\sigma^\#}\text{-}ext\left(D_{\sigma^\#} - ext(P)\right)$. But $D_{\sigma^\#}\text{-}int(P) \subset D_{\sigma^\#}\text{-}ext\left(D_{\sigma^\#} - ext(P)\right)$.
6. $D_{\sigma^\#}\text{-}ext(P \cup B) = D_{\sigma^\#}\text{-}int(P \cup B)^c = D_{\sigma^\#}\text{-}int(P^c \cap B^c)$ [by demorgan law]
   $D_{\sigma^\#}\text{-}ext(P \cup B) = D_{\sigma^\#}\text{-}int(P)^c \cap D_{\sigma^\#}\text{-}int(B)^c$ by(4) of theorem(3.18)
   $D_{\sigma^\#}\text{-}ext(P \cup B) = D_{\sigma^\#}\text{-}ext(P) \cap D_{\sigma^\#}\text{-}ext(B)$

**Remark (3.25)**
1. The converse of $[D_{\sigma^\#}\text{-}ext(P) \subset P^c]$ is not true in general.
2. The converse of [If $P \subset B$, then $D_{\sigma^\#}\text{-}ext(B) \subset D_{\sigma^\#}\text{-}ext(P)$] is not true in general.
3. The converse of $[D_{\sigma^\#}\text{-}ext(P) \subset D_{\sigma^\#}\text{-}ext\left(D_{\sigma^\#} - ext(P)\right)]$ is generally false.

**Definition (3.26):** A point $x$ of a $q$-toplogical space $(X, \tau_q)$, is said to be $D_{\sigma^\#}$-boundary point of a subset $P$ of $X$ iff its neither $D_{\sigma^\#}$-interior nor $D_{\sigma^\#}$exterior point of $P$. The set of all $D_{\sigma^\#}$-boundary points of $P$ is denoted by $D_{\sigma^\#}\text{-}b(P)$.

**Example (3.27):** Let $X = \{a, m, c\}$, where $\tau_1 = \{X, \emptyset, \{a\}, \{m\}, \{a, m\}\}$ , $\tau_2 = \{X, \emptyset, \{a\}\}$, $\tau_2 clg^* = \{X, \emptyset, \{m\}, \{c\}, \{a, m\}, \{a, c\}, \{m, c\}\}$, $\tau_3 = \{X, \emptyset, \{c\}, \{a, c\}, \{m, c\}\}$, $\tau_4 = \{X, \emptyset, \{m\}, \{a, m\}\}$, $\tau_4 clg^* = \{X, \emptyset, \{c\}, \{a, c\}, \{m, c\}\}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}\text{-}O(X) = \{X, \emptyset, \{a\}, \{m\}, \{a, m\}\}$. If we take $P = \{c\}$, $D_{\sigma^\#}\text{-}int(\{c\}) = \emptyset$, $D_{\sigma^\#}\text{-}ext(\{c\}) = D_{\sigma^\#}\text{-}int(\{a, m\}) = \{a, m\}$. Therefore $D_{\sigma^\#}\text{-}b(c) = \{c\}$

**Theorem (3.28):** Let $(X, \tau_q)$ be a $q$-topological space and $P \subset X$, then the point $x$ in $X$ is $D_{\sigma^\#}$-boundary point of $P$ iff every $D_{\sigma^\#}\text{-}nhd$ of $x$ intersects both $P$ and $P^c$

Proof : We have $x \in D_{\sigma^\#}\text{-}b(P)$ iff $x \notin D_{\sigma^\#}\text{-}int(P)$, and $x \notin D_{\sigma^\#}\text{-}ext(P) = D_{\sigma^\#}\text{-}int(P^c)$ iff neither $P$ nor $P^c$ is $D_{\sigma^\#}\text{-}nhd$ of $x$. Iff no $D_{\sigma^\#}\text{-}nhd$ of $x$ can be containd in $P$ or $P^c$.

**Theorem (3.29):** Let $(X, \tau_q)$ be a $q$-topological space and $P \subset X$ , then $D_{\sigma^\#}\text{-}int(P)$, $D_{\sigma^\#}\text{-}ext(P)$, and $D_{\sigma^\#}\text{-}b(P)$ are disjoint and $X = D_{\sigma^\#}\text{-}int(P) \cup D_{\sigma^\#}\text{-}ext(P) \cup D_{\sigma^\#}\text{-}b(P)$. Moreover $D_{\sigma^\#}\text{-}b(P)$ and $D_{\sigma^\#}$-closed set.

Proof : By definition (3.20) $D_{\sigma^\#}\text{-}int(P^c) = D_{\sigma^\#}\text{-}ext(P)$, $D_{\sigma^\#}\text{-}int(P) \subset P$ and $D_{\sigma^\#}\text{-}int(P^c) \subset P^c$. Given that $P \cap P^c = \emptyset$, it follows that $D_{\sigma^\#}\text{-}int(P) \cap D_{\sigma^\#}\text{-}ext(P^c) = D_{\sigma^\#}\text{-}int(P) \cap D_{\sigma^\#}\text{-}int(P^c) = \emptyset$

By definition of $D_{\sigma^\#}$-boundary we have; $x \in D_{\sigma^\#}\text{-}b(P)$ iff $x \in D_{\sigma^\#}\text{-}int(P)$, and $x \notin D_{\sigma^\#}\text{-}ext(P)$, iff $x \notin D_{\sigma^\#}\text{-}int(P) \cup D_{\sigma^\#}\text{-}ext(P)$, iff $x \in \left(D_{\sigma^\#} - int(P) \cup D_{\sigma^\#} - ext(P)\right)^c$, thus

$$D_{\sigma^\#}\text{-}b(P) = \left(D_{\sigma^\#} - int(P) \cup D_{\sigma^\#} - ext(P)\right)^c \tag{2.1}$$

It follows that $D_{\sigma^\#}\text{-}b(P) \cap D_{\sigma^\#}\text{-}int(P) = \emptyset$, and $D_{\sigma^\#}\text{-}b(P) \cap D_{\sigma^\#}\text{-}ext(P) = \emptyset$, and $X = D_{\sigma^\#}\text{-}int(P) \cup D_{\sigma^\#}\text{-}ext(P) \cup D_{\sigma^\#}\text{-}b(P)$
Given that $D_{\sigma^\#}\text{-}int(P)$, and $D_{\sigma^\#}\text{-}ext(P)$ are $D_{\sigma^\#}$-open set, we see from (2.1) that $D_{\sigma^\#}\text{-}b(P)$ is $D_{\sigma^\#}$-closed set.

**Definition (3.30):** A subset $P$ of a $q$-toplogical space $(X, \tau_q)$, then the $D_{\sigma^\#}$-closure of $P$ is The intersection of all sets that are $D_{\sigma^\#}$-closed under the topology and contain P, and indicated by $D_{\sigma^\#}\text{-}cl(P)$, that is $D_{\sigma^\#}\text{-}cl(P)$ is the smallest $D_{\sigma^\#}$-closed set containing $P$.

**Example (3.31):** Let $X = \{a, m, c, d\}$, where $\tau_1 = \{X, \emptyset, \{a\}, \{a, d\}, \{m, c\}, \{a, m, c\}\}$ , $\tau_2 = \{X, \emptyset, \{c\}, \{c, d\}\}$, $\tau_2 clg^* = \begin{Bmatrix} X, \emptyset, \{a\}, \{m\}, \{a, m\}, \{a, c\}, \{a, d\}, \{m, c\}, \{m, d\} \\ \{a, m, c\}, \{a, m, d\}, \{m, c, d\}, \{a, c, d\} \end{Bmatrix}$, $\tau_3 = \{X, \emptyset, \{a\}, \{a, d\}, \{a, m, c\}\}$, $\tau_4 = \{X, \emptyset, \{m, c, d\}\}$, $\tau_4 clg^* = \begin{Bmatrix} X, \emptyset, \{a\}, \{a, m\}, \{a, c\}, \{a, d\}, \{a, m, c\}, \{a, m, d\} \\ \{a, c, d\} \end{Bmatrix}$. Then $(X, \tau_q)$ is $q$-topological space. So $D_{\sigma^\#}\text{-}O(X) = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{m, c\}, \{a, m, c\}, \{m, c, d\}\}$, $D_{\sigma^\#}\text{-}C(X) = \{X, \emptyset, \{a\}, \{d\}, \{a, d\}, \{m, c\}, \{a, m, c\}, \{m, c, d\}\}$. If we take $P = \{c\}$, $D_{\sigma^\#}\text{-}cl(\{c\}) = \{m, c\}$

**Theorems (3.32):** Consider a q-topological space $(X, \tau_q)$ and let $P$ and $B$ be subsets of $X$, subsequently:

1. $D_{\sigma^\#}\text{-}cl(P)$ is the smallest $D_{\sigma^\#}$-closed set contain $P$.
2. $P$ is $D_{\sigma^\#}$-closed iff $D_{\sigma^\#}\text{-}cl(P) = P$

**Proof**

1. This follows from the definition (3.30)
2. If $P$ is $D_{\sigma^\#}$-closed then $P$ its set is the smallest $D_{\sigma^\#}$-closed set containing $P$, and hence $D_{\sigma^\#}\text{-}cl(P) = P$.
   On the other hand, if $D_{\sigma^\#}\text{-}cl(P) = P$, by (1) $D_{\sigma^\#}\text{-}cl(P)$ is $D_{\sigma^\#}$-closed set and so that $P$ is also $D_{\sigma^\#}$-closed.

**Theorems (3.33):** Consider a q-topological space $(X, \tau_q)$ and let P and $B$ be subsets of $X$, subsequently:

1. $D_{\sigma^\#}\text{-}cl(\emptyset) = \emptyset$, $D_{\sigma^\#}\text{-}cl(X) = X$.
2. $P \subset D_{\sigma^\#}\text{-}cl(P)$.
3. If $P \subset B$, then $D_{\sigma^\#}\text{-}cl(P) \subset D_{\sigma^\#}\text{-}cl(B)$.
4. $D_{\sigma^\#}\text{-}cl(P \cup B) = D_{\sigma^\#}\text{-}cl(P) \cup D_{\sigma^\#}\text{-}cl(B)$
5. $D_{\sigma^\#}\text{-}cl(P \cap B) \subset D_{\sigma^\#}\text{-}cl(P) \cap D_{\sigma^\#}\text{-}cl(B)$
6. $D_{\sigma^\#}\text{-}cl(P) = D_{\sigma^\#}\text{-}cl\left(D_{\sigma^\#} - cl(P)\right)$

**Proof**

1. Given that $\emptyset$ is $D_{\sigma^\#}$-closed, where $D_{\sigma^\#}\text{-}cl(\emptyset) = \emptyset$, [by theorem (3.32)], by same way $D_{\sigma^\#}\text{-}cl(X) = X$
2. By theorem (3.32), part (1) hence $P \subset D_{\sigma^\#}\text{-}cl(P)$

3. By (2) $B \subset D_{\sigma^{\#}}\text{-}cl(B)$. Given that $P \subset B$, then $P \subset D_{\sigma^{\#}}\text{-}cl(P)$, then $P \subset D_{\sigma^{\#}}\text{-}cl(B)$, but $D_{\sigma^{\#}}\text{-}cl(B)$ is $D_{\sigma^{\#}}$-closed set, thus $D_{\sigma^{\#}}\text{-}cl(B)$ is $D_{\sigma^{\#}}$-closed set containing $P$, also $D_{\sigma^{\#}}\text{-}cl(P)$ is smallest $D_{\sigma^{\#}}$-closed set containing $P$, hence $D_{\sigma^{\#}}\text{-}cl(P) \subset D_{\sigma^{\#}}\text{-}cl(B)$
4. Given that $P \subset P \cup B$, and $B \subset P \cup B$, then by (3) above, we have -$D_{\sigma^{\#}}\text{-}cl(P) \subset D_{\sigma^{\#}}\text{-}cl(P \cup B)$, and $D_{\sigma^{\#}}\text{-}cl(B) \subset D_{\sigma^{\#}}\text{-}cl(P \cup B)$, hence $D_{\sigma^{\#}}\text{-}cl(P) \cup D_{\sigma^{\#}}\text{-}cl(B) \subset D_{\sigma^{\#}}\text{-}cl(P \cup B)$ (2.2)
   Given that $D_{\sigma^{\#}}\text{-}cl(P)$, and $D_{\sigma^{\#}}\text{-}cl(B)$, are $D_{\sigma^{\#}}$-closed set, also $D_{\sigma^{\#}}\text{-}cl(P) \cup D_{\sigma^{\#}}\text{-}cl(B)$is $D_{\sigma^{\#}}$-closed set by(2), $P \subset D_{\sigma^{\#}}\text{-}cl(P)$, $B \subset D_{\sigma^{\#}}\text{-}cl(B)$. This implies that $P \cup B \subset D_{\sigma^{\#}}\text{-}cl(P) \cup D_{\sigma^{\#}}\text{-}cl(B)$, thus $D_{\sigma^{\#}}\text{-}cl(P) \cup D_{\sigma^{\#}}\text{-}cl(B)$ is $D_{\sigma^{\#}}$-closed set containing $P \cup B$, as well as $D_{\sigma^{\#}}\text{-}cl(P \cup B)$ is smallest $D_{\sigma^{\#}}$-closed set containing $P \cup B$. Therefore $D_{\sigma^{\#}}\text{-}cl(P \cup B) \subset D_{\sigma^{\#}}\text{-}cl(P) \cup D_{\sigma^{\#}}\text{-}cl(B)$ (2.3).
   From (2.2) and (2.3) we get $D_{\sigma^{\#}}\text{-}cl(P \cup B) = D_{\sigma^{\#}}\text{-}cl(P) \cup D_{\sigma^{\#}}\text{-}cl(B)$
5. Given that $P \cap B \subset P$, then $D_{\sigma^{\#}}\text{-}cl(P \cap B) \subseteq D_{\sigma^{\#}}\text{-}cl(P)$ by part(3), and $P \cap B \subset B$, then $D_{\sigma^{\#}}\text{-}cl(P \cap B) = D_{\sigma^{\#}}\text{-}cl(B)$. Hence $D_{\sigma^{\#}}\text{-}cl(P \cap B) \subset D_{\sigma^{\#}}\text{-}cl(P) \cap D_{\sigma^{\#}}\text{-}cl(B)$
6. Given that $D_{\sigma^{\#}}\text{-}cl(P)$ is $D_{\sigma^{\#}}$-closed set, we have [by theorem (3.32), part (2)], we have $D_{\sigma^{\#}}\text{-}cl(P) = D_{\sigma^{\#}}\text{-}cl\left(D_{\sigma^{\#}} - cl(P)\right)$

**Remark (3.34)**
1. The converse of [$P \subset D_{\sigma^{\#}}\text{-}cl(P)$] is not true in general.
2. The converse of [If $P \subset B$, then $D_{\sigma^{\#}}\text{-}cl(P) \subset D_{\sigma^{\#}}\text{-}cl(B)$] is not true in general.

**Conclusion**
In this study, the concept of open sets of type $D_{\sigma^{\#}}$ was clarified, as well as the relationships between the sets derived from them and the observations and proofs based on them. We also propose, as future studies, to find a definition of $D_{\sigma^{\#}}$-continuous and some types of continuous on it, $D_{\sigma^{\#}}$-Homeomorphism, $D_{\sigma^{\#}}$-open space and -closed space, $D_{\sigma^{\#}}$-compact, and $D_{\sigma^{\#}}$-connected based on $D_{\sigma^{\#}}$-topology.

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