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Use of concomitants of record values in the estimation of parameters μ_2 and σ_2 involved in morgenstern type bivariate exponential distribution

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Abstract
In this article we have derived the best linear unbiased estimators (BLUE's) of the parameters μ_2 and σ_2 involved in the Morgenstern type bivariate exponential distribution (MTBED) by concomitants of record values. Efficiency comparisons are also made on the proposed estimators with that of competing unbiased estimators for μ_2 and σ_2 involved in MTBED.

Keywords: Morgenstern family of distributions, Morgenstern type bivariate exponential distribution, best linear unbiased estimation, concomitants of record values

1. Introduction

This section includes some fundamental findings that are pertinent to the current investigation. A quick review of literature on record values, concomitants of record values are recited. An introduction to the Morgenstern family of distributions is also displayed in this section.

The study of record values in many ways parallels to the study of order statistics, indeed they are inextricably related. In any physical phenomena, occurrence of extreme situations attracts the attention of the experimenter. Record value theory developed in statistics; enter forcibly into the study of such situations. In these situations record values and record breaking data are also found useful to draw inferences about the underlying population distribution. Record values and associated statistics are of great importance in several real life problems involving weather, economics and sports data. The statistical study of record values started with Chandler (1952) [8] and now it has spread in different directions. Resnick (1973) [14] and Shorock (1973) [16] documented the asymptotic theory of records. Glick (1978) [9] provides a survey of literature of records. For a detailed discussion on the developments in the theory and application of record values, see Arnold *et al*. (1998) [4], Nagaraja (1988) [12] Nevzorov and Balakrishnan (1998) and Ahsanullah (1995) [13].

In the bivariate set up, concomitants of order statistics and concomitants of record values assume their importance in place of the order statistics and the record values arising from a univariate population. Development of theory and applications of concomitants of record values also provide a very rich source of information on bivariate distributions which further helps to generate methods for the analysis of data arising from them. For a detailed description on the theory of concomitants of record values, see Ahsanullah (1994) [1]. For some latest developments in the theory and application on concomitants of record values, see Ahsanullah and Nevzorov (2000) [3] and Chacko and Thomas (2006) [5].

Let (X, Y) be a bivariate random variables with cumulative distribution function (cdf) F(x, y), which is absolutely continuous and let the corresponding joint pdf be denoted by f(x, y). Let $(X_1, Y_1), (X_2, Y_2) \ldots$ be a sequence of independent observations arising from the distribution f(x, y). Suppose the upper (or lower) record values corresponding to the first component variate X in the sequence $(X_i, Y_i), i = 1,2,\ldots$ alone are available. Then in the available literature, methods are developed to utilize the information contained in these record values to deal with the inference problems associated with random variable X (see Gulati and Padgett, 2003) [10]. But there are situation such as in animals or human beings where measurements on the units of a study variate Y is expensive or painful, but the measurement on an auxiliary variable X which is correlated with Y is very easy and the main interest lies on the study of those units having probably large (or Small) values on Y. In this case the study can be conducted by generating a sequence of independent X observations on the units, constructing the associated sequence of record values and then making the measurement of Y on those units which have record values on X. Then the resulting observations made on Y are known as concomitants of record values. A detailed account of applications of concomitants of record value in inference problems are available in Chacko and Thomas (2008) [6].

1.1 Record value

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables having a common distribution function F(x) which is absolutely continuous and probability density function(pdf) f(x).An observation X_j will be called an upper record value (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i, \forall i < j$. The first observation X_1 is taken as the initial record R_1. The next record R_2 is the observation following R_1 which is greater than R_1 and so on. The records $R_1, R_2, \ldots$ as defined above are sometimes referred to as the sequence of upper records. Similarly, an observation X_j will be called a lower record value if its value is less than that of all previous observations. If we write R_n to denote the n^{th} upper record value then its pdf is given by

$$f_{R_n}(x) = \frac{1}{(n-1)!}[-\log\{1-F(x)\}]^{n-1} f(x) \quad ,-\infty < x < \infty \tag{1.1}$$

and the joint pdf of the m^{th} and n^{th} upper record values for $m < n$ is given by

$$f_{R_{m,n}}(x,y) = \frac{1}{(m-1)!(n-m-1)!}[-\log\{1-F_X(x)\}]^{m-1}\frac{f(x)}{1-F(x)}$$

$$\times[-\log\{1-F(y)\}+\log\{1-F(x)\}]^{n-m-1} f(y), \text{ x < y} \tag{1.2}$$

If we write L_n to denote the n^{th} lower record value, then its pdf is given by

$$f_{L_n}(x) = \frac{1}{(n-1)!}[-\log\{1-F(x)\}]^{n-1} f(x), -\infty < x < \infty \ , \tag{1.3}$$

and the joint pdf of the m^{th} and n^{th} lower record values for m < n is given by,

$$f_{L_{m,n}}(x,y) = \frac{1}{(m-1)!(n-m-1)!}[-\log\{1-X(x)\}]^{m-1}\frac{f(x)}{1-F(x)}$$

$$\times[-\log\{1-F(y)\}+\log\{1-F(x)\}]^{n-m-1} f(y) \text{ y < x} \tag{1.4}$$

1.2 Concomitants of Record Values

Suppose $(X_1,Y_1),(X_2,Y_2),\ldots$ is a sequence of bivariate observations from a population. Let R_n denote the n^{th} upper record with respect to the X variable. Then the random variable which occur together as a Y component in the ordered pair with X component equal to R_n is termed as the concomitants of the n^{th} upper record and may be denoted by $Y_{[n]}$. Let $(X_1,Y_1),(X_2,Y_2),\ldots$ be a sequence of observations drawn from a continuous bivariate population with joint pdf h(x, y) and the marginal pdf's $f_X(x)$ and $f_Y(y)$. Let $F_X(x)$ and $F_Y(y)$ be the cumulative distribution function corresponding to the pdf's $f_X(x)$ and $f_Y(y)$ respectively. Suppose $R_1, R_2, \ldots$ is the sequence of upper record values obtained from the observed X values. Then the pdf of the concomitants of the n^{th} upper record $Y_{[n]}$ is given by (see Arnold *et al*, 1998, p.272) [4].

$$f_{Y_{[n]}}(y) = \int_{-\infty}^{\infty} f_{R_n}(x) f_{Y/X}(y/x)dx, \tag{1.5}$$

where

$f_{Y/X}(y/x)$ is the conditional pdf of Y given X=x and $f_{R_n}(x)$ defined by (1.1) represents the pdf of n^{th} upper record value of the X value. Suppose $L_1, L_2, \ldots$ is the sequence of lower record values obtained from the observed X values. Then the pdf of the concomitants of the n^{th} lower record $Y_{[n]}$ is given by

$$f_{Y_{[n]}}(y) = \int_{-\infty}^{\infty} f_{L_n}(x) f_{Y/X}(y/x)dx \ , \tag{1.6}$$

where

$f_{Y/X}(y/x)$ is the conditional pdf of Y given X=x and $f_{L_n}(x)$ is defined (1.3) represents the pdf of n^{th} lower record value of the X values. The joint pdf of concomitants of two upper records $Y_{[m]}$ and $Y_{[n]}$, for m < n is given by

$$f_{y_{[m,n]}}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{R_{m,n}}(x_1, x_2) f_{Y/X}(y_1/x_1) f_{Y/X}(y_2/x_2)dx_1dx_2 \ , \tag{1.7}$$

where

$f_{R_{m,n}}(x_1, x_2)$ is as defined in (1.2) is the joint pdf of concomitants of two lower records, $Y_{(m)}$ and $Y_{(n)}$, for m < n is given by

$$f_{Y_{(m,n)}}(y_1, y_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{L_{m,n}}(x_1, x_2) f_{Y/X}(y_1/x_1) f_{Y/X}(y_2/x_2)dx_1dx_2, \tag{1.8}$$

where

$f_{L_{m,n}}(x_1, x_2)$ is as defined in (1.4) is the joint pdf of m^{th} and n^{th} lower record values of the X values.

Suppose in an experiment the X observations are based on an inexpensive test and the Y observations are based on an expensive test. Then with an objective of observational economy one may go ahead with measuring Y values of only those individuals whose X measurement breaks the previous records. The resulting data output of such an experiment is the sequence $Y_{[n]}$ of concomitants of upper record values. Similarly in some other situations the data of interest is only the sequence $Y_{[n]}$ of concomitants of lower record values. The study of record concomitants was initiated by Houchens (1984) [11]. For a detailed discussion on the distribution theory and properties of concomitants of record values see, Ahsanullah (1994) [1], Arnold *et al* (1998) [4] and Ahsanullah and Nevzorov (2000) [3].

1.3 Morgenstern Family of Distributions

Morgenstern family M of bivariate distribution functions F(x, y) possessing a representation of the form,

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)\{1+\alpha[1-F_X(x)][1-F_Y(y)]\}, \tag{1.9}$$

$F_X(x)$ and $F_Y(y)$ are two univariate distribution functions and associate parameter α constrained to lie in the interval [-1,1].

When

$F_X(x)$ and $F_Y(y)$ are absolutely continuous with corresponding pdf's $f_X(x)$ and $f_Y(y)$ respectively, the joint pdf corresponding to $F_{X,Y}(x, y)$ is given by

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)\{1+\alpha[1-F_X(x)][1-F_Y(y)]\} \tag{1.10}$$

The family M of bivariate distributions with distribution function $F_{x,y}(x, y)$ as given in (1.9) is also called in the literature as Farlie Gumbel Morgenstern (FGM) family of bivariate distributions. Several applications of concomitants of order statistics in estimating the parameters of Morgenstern Family of Distributions (MFD) are elucidated in Chacko and Thomas (2006) [5]. The system provides a very general expression of a bivariate distribution from which members can be derived by substituting expression of any desired set of marginal distributions. Since both the joint cdf and joint pdf are given in terms of marginal cdf and pdf, it is very easy to generate a random sample from a distribution belonging to MFD. Thus the members of this family can be used in simulation studies, especially when weak dependence between variates is of interest.

Important Morgenstern family of distributions is Morgenstern type bivariate logistic distribution and Morgenstern type bivariate exponential distribution. Chacko and Thomas (2006) [5] estimated two parameter of a five parameter Morgenstern type bivariate logistic distribution using concomitants of record values. Chacko and Thomas (2011) [7] estimated one parameter of a three parameter Morgenstern type bivariate exponential distribution using concomitants of record values. Sajeev Kumar and Sumi (2022) [15] estimated two parameters of a five parameter Morgenstern type bivariate exponential distribution using concomitants of order statistics. Hence in this work our aim is to estimate two parameters of a five parameter Morgenstern type bivariate exponential distribution using concomitants of record values. Throughout this paper we assume concomitants of record values mean concomitants of upper record values.

2. Estimation of parameters μ_2 and σ_2 of Morgenstern type bivariate exponential distribution by concomitants of record values

The bivariate random variable (X, Y) is said to have a Morgenstern type bivariate exponential distribution (MTBED) its pdf is given by a

$$f(x,y)=\begin{cases}\frac{1}{\sigma_1\sigma_2}\exp\{-(\frac{x-\mu_1}{\sigma_1})-(\frac{y-\mu_2}{\sigma_2})\} \\ \times[1+\alpha(2\exp\{-\frac{x-\mu_1}{\sigma_1}\}-1)(2\exp\{-\frac{y-\mu_2}{\sigma_2}\}-1)] \qquad ,x>0,y>0,-1\le\alpha\le 1, \\ \qquad\qquad \mu_1>0,\mu_2>0,\sigma_1>0,\sigma_2>0 \\ 0 \qquad otherwise\end{cases} \tag{2.1}$$

The pdf defined in (2.1) is an extension of the pdf of MTBED given in Chacko and Thomas (2011) [7]. Here we have introduced new location parameters μ_1 and μ_2. Clearly the marginal distributions of X and Y variables are univariate exponential distributions. The joint pdf of standard MTBED is obtained by making the substitutions $V=\frac{X-\mu_1}{\sigma_1}$ and $W=\frac{Y-\mu_2}{\sigma_2}$ in (2.1) and it is given by

$$f_{V,W}(v,w)=\exp\{-v-w\}[1+\alpha(2\exp\{-v\}-1)(2\exp\{-w\}-1)] \tag{2.2}$$

Clearly the marginal distributions of each of V and W of (V,W) with pdf defined by (2.2) is standard exponential distributions given by the pdf's

$$f_V(v)=\exp\{-v\} \tag{2.3}$$

and

$$f_W(w)=\exp\{-w\} \tag{2.4}$$

Let $(V_i, W_i), i=1,2,.$ be a sequence of independent observations drawn from (2.2). Let $U_{[n]}^*$ be the concomitants of the n^{th} record value U_n^* arising from (2.2). Then the pdf $f_{[n]}^*(w)$ of $U_{[n]}^*$ and the joint pdf $f_{[n]}^*$ of $U_{[n]}^*$ and the joint pdf $f_{[m,n]}^*(w_1,w_2)$ of $U_{[m]}^*$ and $U_{[n]}^*$ for m< n are obtained from (1.6) and (1.7) respectively and are given by

$$f_{[n]}^*(w)=\exp\{-w\}\{1+\alpha(1-2^{1-n})(1-2\exp\{-w\})\} \tag{2.5}$$

and for m<n

$$\begin{aligned}f_{[m,n]}^*(w_1,w_2)=\exp\{-w_1-w_2\}[1+\alpha(\frac{1}{2^{m-1}}-1)(2\exp\{-w_1\}-1)] \\ +\alpha(\frac{1}{2^{n-1}}-1)(2\exp\{-w_2\}-1)+\alpha^2(\frac{1}{3^m 2^{n-m-2}}-\frac{1}{2^{m-1}}-\frac{1}{2^{n-1}}+1) \\ \times(2\exp\{-w_1\}-1)(2\exp\{-w_2\}-1)] \qquad w_1,w_2\in R\end{aligned} \tag{2.6}$$

Since the marginal distribution of W of (V, W) is standard exponential distribution with pdf defined by (2.4), we have the following

$$\mu=E[W]=1 \tag{2.7}$$

$$\mu^{(2)} = E[W^2] = 2 \quad (2.8)$$

Using (2.7) and (2.8), we have evaluated,

$$\mu_{2:2} = E[W_{2:2}] = \frac{3}{2} \quad (2.9)$$

and

$$\mu_{2:2}^{(2)} = E[W_{2:2}^2] = \frac{7}{2}\ , \quad (2.10)$$

where $W_{2:2}$ is the largest order statistics of a random sample of size 2 drawn from the distribution of marginal random variable X. On substituting (2.7) and (2.8) in (2.9) and (2.10) we get, for $n \geq 1$,

$$E[U_{[n]}^*] = 1 + \frac{\alpha}{2}(1 - 2^{1-n})\ , \quad (2.11)$$

$$E[(U_{[n]}^*)^2] = 2 + \frac{3\alpha}{2}(1 - 2^{1-n}) \quad (2.12)$$

And for m<n

$$E[U_{[m]}^*, U_{[n]}^*] = 1 - \frac{\alpha}{2}[\frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} - 2] + \frac{\alpha^2}{4}[\frac{1}{3^m 2^{n-m-2}} - \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}} + 1] \quad (2.13)$$

Thus the variances and covariances of concomitants of record values of a random sample of size n arising from (2.2 are given by for ($n \geq 1$)),

$$Var[U_{[n]}^*] = 1 + \frac{\alpha}{2}(1 - 2^{1-n}) - \frac{\alpha^2}{4}(1 - 2^{1-n})^2 \quad (2.14)$$

And for m<n

$$Cov[U_{[m]}^*, U_{[n]}^*] = \frac{\alpha^2}{4}[\frac{1}{3^m 2^{n-m-2}} - \frac{1}{2^{m+n-2}}] \quad (2.15)$$

If we write

$$\xi_r = 1 + \frac{\alpha}{2}(1 - 2^{1-n}) \quad , n \geq 1 \quad (2.66)$$

$$\beta_{n,n} = 1 + \frac{\alpha}{2}(1 - 2^{1-n}) - \frac{\alpha^2}{4}(1 - 2^{1-n})^2\ , n \geq 1 \quad (2.17)$$

and

$$\beta_{m,n} = \frac{\alpha^2}{4}[\frac{1}{3^m 2^{n-m-2}} - \frac{1}{2^{m+n-2}}]\ , \text{m<n} \quad (2.18)$$

Then from (2.11), (2.14) and (2.15), we have the following. For $n \geq 1$

$$E[U_{[n]}^*] = \xi_n, \quad (2.19)$$

$$Var[U^*_{[n]}] = \beta_{n,n} \tag{2.20}$$

And for m<n

$$Cov[U^*_{[m]}, U^*_{[n]}] = \beta_{m,n} \tag{2.21}$$

Let $(X_i, Y_i), i = 1,2,\ldots$ be a sequence of independent observations drawn from a population with pdf defined in (2.1). Clearly we have $X_i = \mu_1 + \sigma_1 V_i$ and $Y_i = \mu_2 + \sigma_2 W_i$ for i=1, 2,… Then by using (2.19), (2.20) and (2.21). We have for $n \geq 1$

$$E[U_{[n]}] = \mu_2 + \sigma_2 \xi_n, \tag{2.22}$$

$$Var[U_{[n]}] = \sigma_2^2 \beta_{n,n} , \tag{2.23}$$

and for m<n

$$Cov[U_{[m]}, U_{[n]}] = \sigma_2^2 \beta_{m,n}, \tag{2.24}$$

where $\xi_n, \beta_{m,n}$ and $\beta_{m,n}$ are defined by (2.16), (2.17), and (2.18) respectively. Clearly from (2.16), (2.17) and (2.18) it follows that $\xi_n, \beta_{n,n}$ and $\beta_{m,n}$ are known constants provided α is known.
Now we derive the BLUE of σ_1 and σ_2 involved in (2.1) and is given in the following theorem.

Theorem 2.1
Let $(X_i, Y_i), i = 1,2,..$ be a sequence of independent observations drawn from the population with pdf defined by (2.1). Let $U_{[n]} = (U_{[1]}, U_{[2]}, \ldots, U_{[n]})'$ be the vector of concomitants of record values arising from (2.1). Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)'$ and $G = ((\beta_{i,j}))$ respectively be the mean vector and dispersion matrix of vector of concomitants of record values arising from (2.2). Then the BLUE's of the parameters μ_2 and σ_2 is given by

$$\hat{\mu}_2 = -\frac{\xi' G^{-1}(1\xi' - \xi 1')G^{-1}}{(\xi' G^{-1}\xi)(1' G^{-1} 1) - (\xi' G^{-1} 1)^2} U_{[n]} \tag{2.25}$$

And

$$\hat{\sigma}_2 = \frac{1' G^{-1}(1\xi' - \xi 1')G^{-1}}{(\xi' G^{-1}\xi)(1' G^{-1} 1) - (\xi' G^{-1} 1)^2} U_{[n]} \tag{2.26}$$

Corresponding variances are given by

$$\text{Var } (\hat{\mu}_2) = \frac{(\xi' G^{-1}\xi)\sigma_2^2}{(\xi' G^{-1}\xi)(1' G^{-1} 1) - (\xi' G^{-1} 1)^2} \tag{2.27}$$

And

$$\text{Var } (\hat{\sigma}_2) = \frac{(1' G^{-1} 1)\sigma_2^2}{(\xi' G^{-1}\xi)(1' G^{-1} 1) - (\xi' G^{-1} 1)^2} \tag{2.28}$$

Proof: If $U^{[n]} = (U_{[1]}, U_{[2]}, \ldots, U_{[n]})'$ then by using (2.22) we can write

$$E[U^{[n]}] = \mu_2 1 + \sigma_2 \xi_n, \tag{2.29}$$

Where 1 is a column vector of n ones and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)'$ .Again by using (2.23) and (2.24) the variance-covariance matrix of $U^{[n]}$ is given by

$$D[U^{[n]}] = G\sigma_2^2, \quad (2.30)$$

where $G = ((\beta_{ij}))$. If α involved in ξ and G are known, then (2.29) and (2.30 together defines a generalized Gauss-Markov set up and then by Generalized Gauss-Markov theorem we got the required result.

Remark: 3.1

If the parameter α involved in (2.1) is unknown, then it should be estimated using copula approach. Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ are random sample of size n taken from (2.1). Let the sample correlation coefficient is q. The population correlation coefficient between X and Y involved in (2.1) is $\frac{\alpha}{2}$. Now the estimated value α is

$$\hat{\alpha} = \begin{cases} -1 & if \quad q < -\frac{1}{4} \\ 4q & if \quad -\frac{1}{4} \le q \le \frac{1}{4} \\ 1 & if \quad q > \frac{1}{4} \end{cases} \quad (2.31)$$

3. Computation

We have evaluated the coefficients of the BLUE's $\hat{\mu}_2$ (defined in (2.25)) and $\hat{\sigma}_2$ (defined in (2.26)) for different values of the sample size n and $\alpha = 0.5, 0.75$ are respectively given in Table 1 and Table 2.

To compare the efficiency of our estimator $\hat{\mu}_2$ defined in (2.25), we take the unbiased estimator

$$\mu_2^* = \frac{2[(1+\frac{\alpha}{2}(1-2^{1-n})U_{[1]} - U_{[n]}]}{\alpha(1-2^{1-n})}, \quad (3.1)$$

with variance

$$V(\mu_2^*) = \frac{4[(1+\frac{\alpha}{2}(1-2^{1-n}))^2\beta_{1,1} + \beta_{n,n} - 2(1+\frac{\alpha}{2}(1-2^{1-n})\beta_{1,n})]\sigma_2^2}{[\alpha(1-2^{1-n})]^2} \quad (3.2)$$

Also to compare the efficiency of our estimator $\hat{\sigma}_2$ defined in (2.26), we have take the unbiased estimator

$$\sigma_2^* = \frac{2[U_{[n]} - U_{[1]}]}{\alpha(1-2^{1-n})}, \quad (3.3)$$

with variance

$$V(\sigma_2^*) = \frac{4[\beta_{n,n} + \beta_{1,1} - 2\beta_{1,n}]\sigma_2^2}{[\alpha(1-2^{1-n})]^2},$$

where $U_{[1]}$ and $U_{[n]}$ are respectively the concomitants of first and n^{th} upper record values arising from (2.1). We have evaluated $V_1 = V(\hat{\mu}_2)$, $V_2 = V(\hat{\sigma}_2)$, $V_3 = V(\mu_2^*)$, $V_4 = V(\sigma_2^*)$ and evaluated RE_1 , the relative efficiency of $\hat{\mu}_2$ relative to μ_2^* and RE_2 , the relative efficiency of $\hat{\sigma}_2$ with respect to σ_2^* for n=2(1)10 for $\alpha = 0.5$ is given in Table 3 and for $\alpha = 0.75$ is given in Table 4.

Table 1: Coefficient of $U_{[r]}$ in the BLUE $\hat{\mu}_2$ for n=2(1)10 and for $\alpha = 0.5, 0.75$.

n	α	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2	0.50	9.000	-8.000								
	0.75	6.333	-5.333								
3	0.50	6.722	-1.167	-4.555							
	0.75	4.813	-0.773	-3.040							
4	0.50	5.714	0.325	-1.975	-3.064						
	0.75	4.134	0.239	-1.319	-2.054						
5	0.50	5.177	0.866	-0.964	-1.828	-2.252					
	0.75	3.769	0.610	-0.640	-1.226	-1.514					
6	0.50	4.857	1.122	-0.459	-1.203	-1.568	-1.749				
	0.75	3.551	0.786	-0.298	-0.804	-1.060	-1.176				
7	0.50	4.651	1.263	-0.166	-0.839	-1.168	-1.330	-1.412			
	0.75	3.412	0.885	-0.102	-0.560	-0.785	-0.897	-0.953			
8	0.50	4.511	1.351	0.021	-0.604	-0.909	-1.061	-1.136	-1.174		
	0.75	3.316	0.946	0.025	-0.402	-0.611	-0.715	-0.767	-0.793		
9	0.50	4.412	1.411	0.150	-0.442	-0.731	-0.874	-0.945	-0.981	-0.999	
	0.75	3.248	0.987	0.112	-0.293	-0.491	-0.589	-0.638	-0.663	-0.675	
10	0.50	4.338	1.454	0.243	-0.324	-0.601	-0.738	-0.806	-0.841	-0.858	-0.866
	0.75	3.198	1.017	0.175	-0.214	-0.403	-0.497	-0.544	-0.568	-0.579	-0.585

Table 2: Coefficient of $U_{[r]}$ in the BLUE $\hat{\sigma}_2$ for n=2(1)10 and for $\alpha = 0.5, 0.75$.

n	α	Coefficients									
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2	0.50	-8.000	8.000								
	0.75	-5.333	5.333								
3	0.50	-5.786	1.357	4.429							
	0.75	-3.873	0.953	2.921							
4	0.50	-4.823	-0.068	1.964	2.927						
	0.75	-3.236	0.004	1.307	1.925						
5	0.50	-4.315	-0.580	1.008	1.759	2.128					
	0.75	-2.900	-0.338	0.681	1.160	1.397					
6	0.50	-4.014	-0.820	0.534	1.173	1.486	1.641				
	0.75	-2.701	-0.499	0.369	0.775	0.981	1.075				
7	0.50	-3.822	-0.952	0.261	0.832	1.112	1.250	1.319			
	0.75	-2.574	-0.588	0.190	0.554	0.752	0.821	0.866			
8	0.50	-3.692	-1.034	0.087	0.613	0.871	0.999	1.062	1.094		
	0.75	-2.488	-0.644	0.076	0.411	0.574	0.656	0.697	0.718		
9	0.50	-3.599	-1.090	-0.033	0.463	0.705	0.825	0.885	0.915	0.930	
	0.75	-2.426	-0.681	-0.003	0.312	0.466	0.542	0.581	0.600	0.610	
10	0.50	-3.530	-1.129	-0.120	0.353	0.584	0.699	0.756	0.784	0.798	0.806
	0.75	-2.381	-0.708	-0.060	0.240	0.387	0.460	0.496	0.514	0.524	0.528

Table 3: $V_1 = \sigma_2^2 V(\hat{\mu}_2), V_2 = \sigma_2^2 V(\hat{\sigma}_2)$, $V_3 = \sigma_2^2 V(\mu_2^*), V_4 = \sigma_2^2 V(\sigma_2^*)$, $RE_1 = \frac{V_2}{V_1}$, $RE_2 = \frac{V_4}{V_2}$ for n=2(1)10 and for $\alpha = 0.5$.

n	V_1	V_2	V_3	V_4	RE_1	RE_2
2	150.500	133.667	150.500	133.667	1.000	1.000
3	70.195	57.760	72.537	60.926	1.033	1.055
4	48.103	37.603	55.378	45.259	1.151	1.204
5	38.556	29.074	49.150	39.628	1.275	1.363
6	33.461	24.587	46.456	37.205	1.388	1.513
7	30.358	21.900	45.201	36.077	1.488	1.647
8	28.362	20.142	44.594	35.532	1.572	1.764
9	26.949	18.947	44.295	35.265	1.644	1.864
10	25.912	18.021	44.147	35.132	1.704	1.950

Table 4: $V_1=\sigma_2^2 V(\hat{\mu}_2), V_2=\sigma_2^2 V(\hat{\sigma}_2)$, $V_3=\sigma_2^2 V(\mu_2^*), V_4=\sigma_2^2 V(\sigma_2^*)$, $RE_1=\frac{V_2}{V_1}$, $RE_2=\frac{V_4}{V_2}$, for n=2(1)10 and for $\alpha=0.75$.

n	V_1	V_2	V_3	V_4	RE_1	RE_2
2	71.305	59.889	71.306	59.889	1.000	1.000
3	34.514	25.938	35.571	27.543	1.031	1.062
4	24.227	16.906	27.574	20.515	1.138	1.213
5	19.738	13.085	24.651	17.979	1.249	1.374
6	17.310	11.075	23.382	16.885	1.351	1.525
7	15.875	9.873	22.790	16.376	1.436	1.659
8	14.916	9.087	22.503	16.129	1.509	1.775
9	14.245	8.539	22.362	16.008	1.570	1.875
10	13.753	8.138	22.292	15.949	1.621	1.960

4. Conclusion

Using concomitants of record values arising from Morgenstern type bivariate exponential distribution (MTBED) defined in (2.1), one can estimate BLUE's of the parameters μ_2 and σ_2 involved in (2.1). Also it may be noted that in all the cases the efficiency of our estimators $\hat{\mu}_2$ and $\hat{\sigma}_2$ (BLUE's) are much better than the unbiased estimators μ_2^* and σ_2^* considered for comparison.

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