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Study of numerically effective vector bundles

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Abstract
Parabolic bundles were introduced in [1]. It is now known the various notions related to the usual vector bundles actually extend to be context of parabolic vector bundles. In [2] the notion of ampleness of a vector bundle was extended to the parabolic category. In [3] and [4] various results on usual ample vector bundles were generalized to parabolic ample bundles.

Keywords: effectiveness, ampleness, parabolic, bundles, projective curve, Narasimhan, filtration, tautological, hyperplanes, homomorphism

Introduction
The notion of numerically effectiveness of a vector bundle is very closely related to the notion of ampleness. Defining the notion of numerically effectiveness of a parabolic vector bundle, we generalize some known results on usual numerically effective vector bundles to the more general context of parabolic bundles.

Properties numerically effective vector bundles
Let X be a connected smooth projective variety over C. For a vector bundle E over X, the projective bundle over X consisting of all hyper planes in the fivers of E will be denoted by PE. The tautological relative ample line bundle over PE will be denoted by O (1). We recall the definition of a numerically effective vector bundle.

Definition 1.2.3
A line bundle L over X is called numerically effective (abbreviated as nef) if for any morphism **f: C⌡X,** where C is a connected smooth projective curve, the inequality is valid.

$$.\deg(f^*L) \exists\ 0 \qquad \text{…. 1.1}$$

More generally, a vector bundle E over X is called numerically effective if the line bundle $O_{PE}(1)$ over PE is nef in the above sense. For a vector bundle V over a connected smooth curve C, let $d_{min}(V)$ denote the degree of the final piece of the graded object for the Harder – Narasimhan filtration of V. In other words, if

$$O= V_o\ \delta V_1\ \delta V_2 \ldots\ldots\ \delta\ V_1\ \delta\ V_1{+}_1 = V \qquad \text{…..1.2}$$

is the Harder- Narasimhan filtration of V, then d_{min} (V) :=deg (V/V_1).

Proposition 1.2.3
A vector bundle A over X is nef if and only if for any morphism f form a curve C to X, as in definition 1.1, the inequality Dmin (f*E)∃0 is valid.

Proof:
(If D_{min} (f*E) ∃ 0, then it is easy to see that deg (L) ∃0,
where L is any quotient line bundle of f*E. Indeed, if

$$0 = E_0\ \delta\ E_1\ \delta \ldots\ldots\ \delta\ E_1\ \delta\ E_1{+}_1 = f/E \qquad \text{……..1.3}$$

is the Harder- Narasimhan filtration of f*E and deg (V/V_1)∃ 0, then for any line bundle L' over C with deg(L')< 0,the following is valid

$H^0(C{\cdot}Hom(E_{1+1}/E_1,L') = 0$ 1.4

for all i É [0,1]. This implies that L' cannot be a quotient line bundle of f*E.

The above observation that deg (L) ∃0, immediately yields that the vector bundle E is nef [6]. To prove the converse, we assume that E is nef. Taking f as above, and let a filtration as in (2.3) be the Harder-Narasimhan of f*E. Let r be rank of f*E/E_1.Since E is nef, the pullback f*E is also nef, and hence we have that ^rf*E is nef [Vi 6], But the line bundle ^r (f*E/E_1') is a quotient of $\wedge^r f$*E. Thus the inequality d_{min} (f*E) = deg($\wedge^r$(f*E/E_1))∃ 0 is valid. This completes the proof of the proposition. We fix a polarization L on X. Any vector bundle V over X admits a unique Harder-Narasimhan filtration [11]. As before, define

$d_{min}(V)$: =deg(V/V_1): =□$_x$ $c_1(V/V_1)$∪ $c_1(L)^{d-1}$

where d = dimcX.

Theorem 1.2.5

If E is a nef vector bundle over X, then d_{min}(E)∃ 0. If dincX =1, then any vector bundle E over X, with d_{min}(E)∃ 0, is nef.

Proof:

Let 0 = E_0 δ E_1 δ E_2 δ ....δ E_1 δ E_{1+1} = E be the Harder-Narasimhan filttation of E.

Let f: C!X be a smooth irreducible complete intersection curve such that the restriction of al E_{i+1}/E_i, 0 # 1# 1, to C is a semistion vector bundle. The existence of such a curve C is ensured by the main theorem of [5] which says that a vector bundle over a smooth projective variety is semistable if and only if its restriotion to the general complete intersection curve of sufficiently large degree is semistable. Thus the restriction 0 = E_0 δ E_1/c δ E_2/c δ ......δ E_1/c δ E_{1+1}/c = f*E of the above filtration of E to c is actually the Harder-Narasimhan filtration of f*E. Indeed, if F δ f*E is the maximal semistable subbundle, then μ(F)∃ μ(E_1/c), and hence F does not admit any nonzero homomorphism to the quotient vector bundle (E_{1+1}/c)|(E_1/c) for any i ∃ 1. Thus F must coincide with E_1/c. Now the above claim is established by using induction on the lengh of the Hardr-Narasimhan filtration of the vector bundle E.

If E is nef, then from proposition 1.2.3 it follows that

d_{min}(f*E)∃ 0. This, in turn, implies that d_{min}(E)∃0.

Now let X be a Riemann surface, and let E be a vector bundle over X with **d_{min}(E)∃0.** This implies that d_{min} (S^k(E))∃0, where S^k(E)is the k- fold symmetric tensor power of E.

Let L be a line bundle over X with deg(L)> 0. A consequence of the inequality d_{min} (S^k(E))∃ 0 is that any quotient of S^k(E) is nonnegative degree. Hence any quotient of S^k(E)ρ L is of strictly positive degree. Now from a theorem of [6] we conclude that s^k(E)ρ L is ample. The theorem of [6] in question says that a vector bundle over a complete curve is ample if and only if its degree and also the degrees of all its quotient bundles are all strictly positive. For a non-constant morphism of curves f: C! X, if V is an ample vector bundle over X, then f*V is ample. So from [3] we conclude that E must be nef. This completes the proof of the theorem. We remark that the second part of Theorem 4.2.5 can also be deduced using proposition 4.2.3. A vector bundle E over a projective manifold X is called numerically flat if both E its dual E* are nef [7].

Proposition 1.2.6

A vector bundle E is numerically flat if and only if e is semirically with c_1(E) =0 = c_2(E).

Proof:

Let E be a numerically flat vector bundle over X. Since the Harder-Narasimhan filtration of E* is simply the dual of the Harder-Narasimhan filtration of E, the conditions obtained from Theorem 1.2.5, namely d_{min}(E)∃0 and d_{min}(E*)∃ 0, immediately imply the E is semistable with C_1(E) = 0. That C_2(E) = 0 follows, o f courde, from [7]. Converely, let E be a semistable vector bundle over X with C_1(E) = 0 C_2(E). From [8] we know that E admits a filtration

F_1 δ F_2 δ ..... δ F_k δ F_{k+1} = E

and a flat connection ∉ on E which preserves each F_i and the induced connection on each F_{i+1}/F_i is a unitary flat connection. Thus for any map f: C ! X from a curve C, the vector bundle f*E over C has a flat connection, namely f*∉, such that on each f*F_{i+1}/F_i it induces a unitary flat connection. This implies that f*E is a semistable vector bundle over the curve C with deg (f*E) = 0. Now from the second part of Theorem 4.2.5 we conclude that f*E is nef. Thus E must be nef the same argument shows that E* is nef. This completes the proof of the proposition. In the next section we will define the notion of nefness in in the context of parabolic sheaves.

Parabolic Nef Bundles

Let D be an effective divisor on X. Let E. be a parabolic vector bundle ample ample bundles were defined; this generalizes the notion of ample vector bundles to the parabolic context.

Definition 1.3.1

A parabolic vector bundle E, is called parabolic nef if there is an ample line L over X such that S^k(E)ρ L is parabolic ample for every k, where S^k(E.), denotes the k-fold parabolic symmetric tensor power of the parabolic bundle E [2] for the definition of parabolic tensor product. If the parabolic structure of E, is trivial, i.e., zero is the only parabolic weight, then from proposition 2.9 of [3] it follows that E. is parabolic nef if and only if the underlying vector bundle is nef in the usual sense. Henceforth, we will assume that the parabolic divisor D on X is a normal crossing divisor. By this we mean that D is reduced, each irreducible component of D is smooth, and furthermore, the irreducible components intersect transversally. The parabolic structure of a parabolic bundle E is defined as follows:

For each irreducible component D_i of the parabolic divisor D, a filtration by coherent subsheaves of the voctor bundle $E\backslash_{Di}$ over D_i is given, together with a system of parabolic weights corresponding to the filtration [1, 9].

We will henceforth consider only those parabolic bundles E, for which the filtration over any D_i defininig the quasi-parabolic structure, is by subbundles of $E/_{Di}$. Let E be a parabolic vector bundle with rational parabolic weights. Then there is a Galois covering map p : c ! X and on orbifold vector bundle E with rational parabolic bundle E is obtained by taking invariants of the direct image of the twists of V using the irreducible components of D [9].

Proposition 1.3.2

A parabolic vector bundle E with rational parabolic weights is parabolic nef if and only if the underlying vector bundle for the corresponding orbifold vector bundle V on Y is nef in the usual sense.

Proof:
Let L be an ample line bundle over X. Since the above covering map p is finite, the line bundle p*L over Y is also ample. We assume that the vector bundle V is nef. So $S^{k(V)}\rho$ p*L is ample for sufficiently large k [3]. Since the orbifold bundle $S^k(V)$ corresponds to the parabolic bundle $S^k(E.)$ [2], and furthermore, from the definition of parabolic amplitude it si immediate that the parabolic bundle corresponding to an orbifold bundle whose underlying vector bundle is ample, is actually parabolic ample, we conclude that E. is parabolic nef. Now we assume that E. is parabolic nef, Lemma 4.6 of [2] says that $S^{k(V)}\rho$ L is ample if $S^{k(E)}\rho L$ is parabolic ample. So we conclude that V must be nef. This completes the proof of the proposition. As the tensor product of a nef vector bundle and an ample vector bundle is ample, the above proposition has the following corollary.

Corollary 1.3.3
Let E. and F. be two parabolic vector bundles with ratonal parabolic weights and with parabolic structure over a normal crossing divisor D. We assume that E. is parabolic nef and F. is parabolic ample. Then the parabolic tensor product E. ρ F. is parabolic ample. We fix a polarization over X to define the parabolic degree of a parabolic bundle. A parabolic vector bundle admits a canonical filtration of parabolic subsheaves with each subsequent quotient parabolic *ds* filtration to the parabolic context. Following the definition of d_{min} in section 1.2 we make the following definition.
For a parabolic sheaf E,*define $\mathbf{d_{min}^{par}(E.)}$ to be the parabolic degree of the minimal parabolic semistable subquotient of E. or in other words, d^{par}_{min} (E.) is the parabolic degree of the final piece of the graded object for the Harer-Narasimhan filtration of E.
Now, as for corollary 1.3.3 the proposition 1.3.2 `combines with Theorem 1.2.5 to give the following corollary.

Corollary 1.3.4
Let E_* be a parabolic vector bundle with rational parabolic weights. If E* is parabolic nef, then dpar min (E.)∃ 0. If dim X= 1, then the converse is also true; namely, if the inequlity d^{par}_{min}(E.) ∃ 0 is valid, then E_* must be parabolic nef.
A parabolic vector bundle E. will be called numerically flat it both E_* and its parabolic dual E, are parabolic nef. Let E. be a numerically flat parabolic bundle over X with rational parabolic weights and with parabolic structure over a normal crossing divisor D. Let V!Y be the orbifold bundle corresponding to E. for a suitable Galois covering map. p:Y ! X with Galois group G. Proposition 1.3.2 says that V is numerically flat, i.e., both V and V* are nef. Now proposition 1.2.6 says that v is semistable with $c_1(V) = 0 = c_2(V)$. Since V is semistable, from [9] it follows that E. is parabolic semistable. Since the first and the second chern class of V vanish, from [10] it follows that first and the second parabolic shern class of E. vanish. conversely, if E. is parabolic semistable with its first and the second parabolic chern class zero, then from [9] and [10] we know that the corresponding orbifold bundle V is semistable whit the first and the second shern class of V being zero. So proposition 1.2.6 yields that v is numerically flat. Now proposition 1.3.2 says that the parabole bundle E. is numerically flat. Thus we have proved the following.

Theorem 1.3.5
A parabolic bundle E. with rational parabolic weights is numerically flat if and only if E. is parabolic semistable with vanishing first and second parabolic chern classes. using [10], from the above theorem it is easy to deduced that a parabolic vector bundle E. over X, with rational parabolic weights, is numerically flat if and only if the following condition is valid : the underlying vector bundle E for the parabolic vector bundle E. has a filtration by subbundles of E such that each subsequent, quotient vector bundle with the induced parabolic structure, induced by E. corresponds to a unitary representation of the fundamental group of the complement X-D, where D is the divisor on X over which the parabolic structure of E is defined. The above statement can also be deduced using proposition 1.3.2 together with [7].

Conclusion
Numerical study of effective vector bundles has been presented. Here we defined numerically effective bundles in the parabolic category. Some properties of the usual numerically effective vector bundles are shown to be valid in the more general context of numerically effective parabolic vector bundles. Various properties of effective vector bundles, definitions, theorems and propositions were presented and proved. The parabolic 'negf 'bundles were also presented.

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