International Journal of Statistics and Applied Mathematics 2017; 2(2): 60-63

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2017; 2(2): 60-63

www.mathsjournal.com
Received: 25-01-2017
Accepted: 27-02-2017

Vinita Vijai
Associate Professor,
Department of Mathematics,
Isabella Thoburn College,
Lucknow, Uttar Pradesh, India

Correspondence
Vinita Vijai
Associate Professor,
Department of Mathematics,
Isabella Thoburn College,
Lucknow, Uttar Pradesh, India

A study of convex properties of mean values of entire functions of several complex variables represented by multiple Dirichlet series

Vinita Vijai

Abstract
In this paper, an attempt is made to study some convex problems relating mean values $I_2(\sigma_1,\sigma_2)$ and$m_{2,k}(\sigma_1,\sigma_2)$ of an entire function $f(s_1,s_2)$ of two complex variables.

Keywords: Convex properties, mean values, entire functions, several complex variables represented, multiple Dirichlet series

Introduction

Let us consider

(1.1) $\quad f(s_1,s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(s_1\lambda_m + s_2\mu_n),$

$\left((s_j = \sigma_j + it_j\right), j = 1,2$ where $a_{m,n} \in C,$

the field of complex numbers, $\lambda_m' s, \mu_n' s$ are real, and

$0 < \lambda_1 < \lambda_2 < \cdots \lambda_m \to \infty;$

$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty.$

It has been proved [1] that if

(1.2) $\lim_{m\to\infty} \frac{\log m}{\lambda_m} = 0, \lim_{n\to\infty} \frac{\log n}{\mu_n} = 0,$

The domain of convergence of the series (1.1) coincides with its domain of absolute convergence. Also, Sarkar [2, pp.99] has shown that the necessary and sufficient condition that the series

(1.1) satisfying (1.2) to be entire is that

(1.3) $\lim_{(m,n)\to\infty} \frac{\log|a_{m,n}|}{\lambda_m+\mu_n} = -\infty$

Let the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3) be denoted by F.
Then f$\in F$denotes an entire function over C^2. The results can be extended to several complex variables.
Corresponding to an f$\in F$, the maximum modulus M $=M_f$ and the maximum term $\mu = \mu_f$ on R^2 Are defined as [2, pp100]

$M(\sigma) = M_f(\sigma_1, \sigma_2) = \max\{|f(s_1, s_2)|: s_1, s_2 \in C, \operatorname{Re} s_1 = \sigma_1, \operatorname{Re} s_2 = \sigma_2\}$

*w*here N is the set of natural numbers.

The mean value $I_2(\sigma_1, \sigma_2)$ of $|f(s_1, s_2)|$ is defined as

$$(1.4) I_2(\sigma_1, \sigma_2; f) = I_2(\sigma_1, \sigma_2) = \lim_{T_1 T_2 \to \infty} \frac{1}{4T_1T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^2 dt_1\, dt_2$$

Now we define the mean value $m_{2,k}(\sigma_1, \sigma_2)$ of $|f(s_1, s_2)|$ as

$$(1.5)\ m_{2,k}(\sigma_1, \sigma_2; f) = m_{2,k}(\sigma_1, \sigma_2)$$

$$= \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2 e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} \{|f(x_1 + it_1, x_2 + jt_2)|^2 e^{kx_1} e^{kx_2}\} dx_1 dx_2 dt_1 dt_2$$

Where k is any positive number.
These mean values defined by (1.4) and (1.5) are increasing functions of σ_1 and σ_2
For

$$|f(s_1, s_2)|^2 = f(s_1, s_2)\overline{f(s_1, s_2)}$$

$$= \sum_{m_1 n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1 \lambda m + \sigma_2 \mu_n)\}$$

$$+ \sum_{m \neq M} \sum_{n \neq N} a_{m,n} \bar{a}_{M,N} \exp\{\sigma_1(\lambda_m + \lambda_M) + \sigma_2(\mu_n + \mu_N) + it_1(\lambda_m - \lambda_M) + it_2(\mu_n - \mu_N)\}$$

The series on the right being absolutely convergent, the resulting series is uniformly convergent for any finite t_1 and t_2 range, therefore integrating term by term, the terms for which m$\neq M$, n$\neq N$, vanish as $T_1, T_2 \to \infty$ and we get

$$(1.6) I_2(\sigma_1, \sigma_2) = \sum_{mn=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma, \lambda_m + \sigma_2 \mu_n)\}$$

Hence $I_2(\sigma_1, \sigma_2)$ is an increasing function of σ_1 and σ_2. Also from (1.4) and (1.5)

$$(1.7)\ m_{2,k}(\sigma_1, \sigma_2) = \frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} I_2(x_1, x_2) e^{kx_1} e^{kx_2} dx_1\, dx_2$$

Using (1.6) we obtain

$$m_{2,k}(\sigma_1, \sigma_2) = \frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \int_0^{\sigma_1} \int_0^{\sigma_2} [\sum_{m,n=1}^{\infty} |a_{m,n}|^2 \exp\{2(\sigma_1 \lambda_m + \sigma_2 \mu_n)\} e^{kx_1} e^{kx_2} dx_1\, dx_2]$$

$$= 4 \sum_{m,n=1}^{\infty} \frac{|a_{m,n}|^2 \left(e^{2\lambda} m^{\sigma_1} - e^{-k\sigma_1}\right)\left(e^{2\mu_n \sigma_2} - e^{-k\sigma_2}\right)}{(2\lambda_m + k)(2\mu_n + k)}$$

Thus $m_{2,k}(\sigma_1, \sigma_2)$ is also an increasing function of $\sigma_1 and\ \sigma_2$.

We now prove following Theorems.

Theorem 1

$Log\ I_2(\sigma_1, \sigma_2)$ is a convex function of σ_1 for a fixed value of σ_2 and
Vice-versa.

Proof
Let I_2', I_2''denote partial derivatives of I_2 with respect to σ_1. Then

$$\frac{\partial^2}{\partial \sigma_1^2}(\log I_2) = \frac{I_2 I_2'' - I_2' 2}{I_2^2},$$

And by Schwarz's Inequality

$I_2'^2 \;\; = \left[\Sigma\left|a_{m,n}\right|^2 2\lambda_m \exp\{2(\sigma_1\lambda_m + \sigma_2\mu_n)\}\right]$

$\leq [\sum\left|a_{m,n}\right|^2 \exp[2\{\sigma_1\lambda_m + \sigma_2\mu_n\}]X$

$[\sum\left|a_{m,n}\right|^2 4\lambda_m{}^2 \exp[2\{\sigma_1\lambda_m + \sigma_2\mu_n\}]$

$=I_2 I_2''$

Hence the result and vice-versa.

Theorem 2

$e^{k\sigma_1} I_2(\sigma_1,\sigma_2)$ is a convex function of $e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)$ for a fixed σ_2

Proof
Using (1.7) we have

$$\frac{\partial\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\partial\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}} = \frac{\frac{\partial}{\partial\sigma_1}\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\frac{\partial}{\partial\sigma_1}\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}}$$

$$=\frac{ke^{k\sigma_1} I_2(\sigma_1,\sigma_2) + e^{k\sigma_1}\frac{\partial}{\partial\sigma_1} I_2(\sigma_{11}\sigma_2)}{\frac{\partial}{\partial\sigma_1}\left\{\frac{4}{e^{k\sigma_1} e^{k\sigma_2}} \cdot e^{k\sigma_1}\int_0^{\sigma_1}\int_0^{\sigma_2} I_2(x_1,x_2)\{e^{kx_1} e^{kx_2}\, dx_1\, dx_2\right\}}$$

$$=\frac{ke^{k\sigma_1} I_2(\sigma_1,\sigma_2) + e^{k\sigma_1}\frac{\partial}{\partial\sigma_1} I_2(\sigma_1,\sigma_2)}{\frac{4}{e^{k\sigma_2}} \cdot e^{k\sigma_1}\int_0^{\sigma_2} I_2(\sigma_1,x_2) e^{kx_2}\, dx_2}$$

Since $I_2(\sigma_1,x_2)e^{kx_2}$ is an increasing function of σ_1, x_2 and therefore applying the second mean value theorem, we have

$$\frac{\partial\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\partial\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}} = \frac{ke^{k\sigma_1} I_2(\sigma_1,\sigma_2) + e^{k\sigma_1}\frac{\partial}{\partial\sigma_1} I_2(\sigma_1,\sigma_2)}{\frac{4}{e^{k\sigma_2}} e^{k\sigma_1} I_2(\sigma_1,\sigma_2) e^{k\sigma_2}\int_{\xi}^{\sigma_2} dx_2}$$

Where $0 < \xi < \sigma_2$

$$=\frac{kI_2(\sigma_1,\sigma_2) + \frac{\partial}{\partial\sigma_1} I_2(\sigma_1,\sigma_2)}{4I_2(\sigma_1,\sigma_2)(\sigma_2-\xi)}$$

$$=\frac{1}{4(\sigma_2-\xi)}\left\{k + \frac{\left(\frac{\partial}{\partial\sigma_1}\right)(I_2(\sigma_1,\sigma_2))}{I_2(\sigma_1,\sigma_2)}\right\}$$

The right hand side increases with σ_1 for a fixed σ_2, follows from Theorem1.

Therefore

$\frac{\partial^2\{e^{k\sigma_1} I_2(\sigma_1,\sigma_2)\}}{\partial^2\{e^{k\sigma_1} m_{2,k}(\sigma_1,\sigma_2)\}} > 0$, for a fixed σ_2. Hence the result

Theorem 3

$e^{k\sigma_2} I_2(\sigma_1,\sigma_2)$ is a convex function of $e^{k\sigma_2} m_{2,k}(\sigma_1,\sigma_2)$ for a fixed σ_1.

The proof is similar to Theorem 2.

Theorem 4

$Log\, m_{2,k}(\sigma_1,\sigma_2)$ is a convex function of σ_1 for a fixed value of σ_2 and vice-versa

Proof

Using (1.7), we have

$$\frac{\partial\{log\, m_{2,k}(\sigma_1,\sigma_2)\}}{\partial\sigma_1}$$

$$= \frac{\partial}{\partial\sigma_1}\{\log m_{2,k}\ (\sigma_{1\,,}\sigma_2)\}$$

$$= \frac{4}{m_{2,k}(\sigma_1,\sigma_2)}[\frac{1}{e^{k\sigma_2}}\{-ke^{-k\sigma_1}\int_0^{\sigma_1}\int_0^{\sigma_2} I_2(x_1,x_2)e^{kx_1+kx_2}\,dx_1\,dx_2 + \frac{1}{e^{k\sigma_1}}e^{k\sigma_1}\int_0^{\sigma_2} I_2(\sigma_{1,}x_2)e^{kx_2}\,dx_2\}]$$

$$=\frac{1}{m_{2,k}(\sigma_1,\sigma_2)}\{-km_{2,k}(\sigma_1,\sigma_2) + \frac{4}{e^{k\sigma_2}}\int_0^{\sigma_2} I_2(\sigma_1,x_2)e^{kx_2}\,dx_2\}$$

Since $I_2(\sigma_1,x_2)e^{kx_2}$ is an increasing function of σ_1, x_2 and therefore applying the second mean-value theorem, we get

$$\frac{\partial\{log\, m_{2,k}(\sigma_1,\sigma_2)\}}{\partial(\sigma_1)}$$

$$=\frac{1}{m_{2,k}(\sigma_1,\sigma_2)}\left\{-km_{2,k}(\sigma_1,\sigma_2) + \frac{4I_2(\sigma_1,\sigma_2)e^{k\sigma_2}}{e^{k\sigma_2}}\int_\xi^{\sigma_2} dx_2\right\},\ 0< \xi < \sigma_2$$

$$=\frac{1}{m_{2,k}(\sigma_1,\sigma_2)}\{-km_{2,k}(\sigma_1,\sigma_2) + 4(\sigma_2-\xi)I_2(\sigma_1,\sigma_2)\}$$

$$=\left\{4(\sigma_2-\xi)\frac{I_2(\sigma_1,\sigma_2)}{m_{2,k}(\sigma_1,\sigma_2)} - k\right\}$$

The right-hand side increases with σ_1 for a fixed σ_2, follows from Theorem

Therefore,

$\frac{\partial^2\{log\, m_{2,k}(\sigma_1,\sigma_2)\}}{\partial\sigma_1^2}>0$, for a fixed value of σ_2. Hence the result and vice-versa.

References

1. Janusauskas AI. Elementary theorem on the convergence of doubleDirichlet Series – Dokl. Akad. Nauk SSSR. 1977;234:1.
2. Sarkar PK. On order and type of an entire function in several complex variables represented by Dirichlet Series – Bull. Cal. Math. Soc. 1982;74:99-11.